Secure Computation

Unconditionally Secure Multi-Party Computation

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Overview

- "Completeness theorems for non-cryptographic fault-tolerant distributed computation"
  - M. Ben-Or, S. Goldwasser, A. Wigderson, 1988.
  - Published concurrently with "Multiparty unconditionally secure protocols" Chaum, Crepau, Damgard.
- Published after the results of Yao and GMW, with the motivation of obtaining results without any intractability assumptions.

Overview

- "Completeness theorems for non-cryptographic fault-tolerant distributed computation"
  - M. Ben-Or, S. Goldwasser, A. Wigderson, 1988.

- The setting
  - A complete synchronous network of n parties
  - Each party P_i has an input x_i
  - Communication channels between parties are secure
  - The solution for the malicious case requires a broadcast channel

Overview (contd.)

- The function $f(x_1,\ldots,x_n)$ is represented by an arithmetic circuit over a field F (say, modulo a large prime)
  - Contains addition and multiplication gates in F
  - Can be more compact than a Boolean circuit
  - We need only care about deterministic functionalities:
  - A randomized functionality $f(r; x_1,\ldots,x_n)$ can be computed by each party providing (r_i,x_i), and the circuit computing and using $r=r_1\oplus\ldots\oplus r_n$.

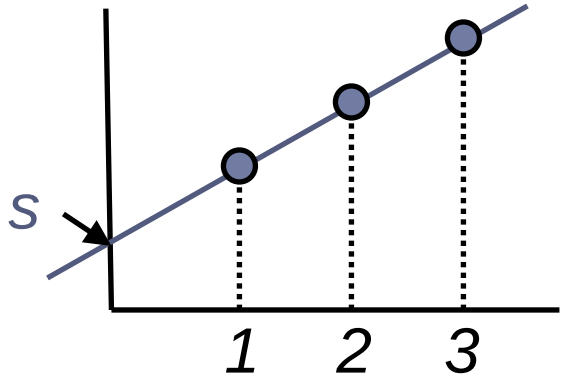
Overview (contd.)

- The construction provides unconditional security
  - Against semi-honest adversaries controlling t<n/2 parties
  - Against malicious adversaries controlling t<n/3 parties

- Unlike the GMW construction, which is based on cryptographic assumptions
  - oblivious transfer
  - ZK proofs

Main tool – secret sharing

- t-out-of-n secret sharing
- Given a secret s, provide shares to n parties, s.t.
  - Any *t* shares enable the reconstruction of the secret
  - Any *t-1* shares reveal nothing about the secret
- Consider *2*-out-of-*n* secret sharing.
  - Define a line which intersects the Y axis at *S*
  - The shares are points on the line
  - Any two shares define *S*
  - A single share reveals nothing



t-out-of-n secret sharing

- Fact: Let F be a field. Any d+1 pairs (a_i , b_i) define a unique polynomial P of degree ≤ d, s.t. $P(a_i)=b_i$. (assuming d < |F|).

- Shamir's secret sharing scheme:
  - The secret S is an element in a field (say, in Zp).
  - Define a polynomial P of degree t-1 by choosing random coefficients $a_1,\ldots,a_{t-1}$ and defining

    $P(x) = a_{t-1}x^{t-1}+\ldots+a_1x+\underline{S}$.
  - The share of party P_j is $(j, P(j))$.

t-out-of-n secret sharing

- Reconstructing the secret:
  - Assume we have $P(x_1),\ldots,P(x_t)$.
  - Use Lagrange interpolation to compute the unique polynomial of degree ≤ t-1 which agrees with these points.
  - Output the free coefficient of this polynomial.

- Lagrange interpolation
  - $P(x) = \sum_{i=1..t} P(x_i)\cdot L_i(x)$
  - where $L_i(x)=\prod_{j\neq i}(x-x_j) / \prod_{j\neq i}(x_i-x_j)$

    (Note that $L_i(x_i)=1$, $L_i(x_j)=0$ for $j\neq i$.)

Properties of Shamir's secret sharing

- Perfect secrecy: Any t-1 shares give no information about the secret, $\Pr(\text{secret}=s \mid P(1),\ldots, P(t-1)) = \Pr(\text{secret}=s)$.
- Proof:
  - Intuition from 2-out-of-n secret sharing:
  - The polynomial is generated by choosing a random coefficient a and defining $P(x)= a \cdot x+s$.
  - Suppose that the adversary knows the share $P(1)=a \cdot 1+s$.
  - For any value of s, there is a one-to-one correspondence between a and $P(1)$ ($a=P(1)-s$).
  - Since a is uniformly distributed, so is $P(1)$
    - Therefore $P(1)$ does not reveal any information about s.

Properties of Shamir's secret sharing

- Perfect secrecy: Any t-1 shares give no information about the secret.
- Proved by showing that, even given S, any t-1 shares are uniformly distributed.
- Proof:
  - The polynomial is generated by choosing a random polynomial of degree t-1, subject to P(0)=S.
  - Suppose that the adversary knows the shares P(1),…,P(t-1).
  - The values of P(1),…,P(t-1) are defined by an invertible set of t-1 linear equations of $a_1,\ldots,a_{t-1}$, s.
    - $P(i) = \Sigma_{j=1,\ldots,t-1} (i)^j a_j + s.$

Properties of Shamir's secret sharing

- Proof (cont.):
  - The values of P(1),…,P(t-1) are defined by an invertible set of t-1 linear equations of $a_1,\ldots,a_{t-1}$, s.
    - $P(x_i) = \Sigma_{j=1,\ldots,t-1} (i)^j a_j + s$.
  - For any possible value of s, there is a exactly one set of values of $a_1,\ldots,a_{t-1}$ which gives the values P(1),…,P(t-1).
    - This set of $a_1,\ldots,a_{t-1}$ can be found by solving a linear system of equations.
  - Since $a_1,\ldots,a_{t-1}$ are uniformly distributed, so are the values of $P(x_1),\ldots,P(x_{t-1})$.
    - $\Rightarrow P(x_1),\ldots,P(x_{t-1})$ reveal nothing about s.

Additional properties of Shamir's secret sharing

- Ideal size:
  - Each share is the same size as the secret.

- Homomorphic property:
  - Suppose P(1),…,P(n) are shares of S,
    and P'(1),…,P'(n) are shares of S',
    then P(1)+P'(1),… ,P(n)+P'(n) are shares of S+S'.

The BGW protocol

- Input sharing phase
- Computation phase
- Output reconstruction phase

- Main idea:
  - for every wire, the parties will know a secret sharing of the value which passes through that wire.

BGW protocol – input phase

- Let $t<n/2$ be a bound on the number of corrupt parties.
- Each P_i generates a $(t+1)$-out-of-n sharing of its input x_i.
  - Namely, chooses a polynomial $f_i()$ of degree t over F, s.t. $f_i(0)=x_i$
  - Any subset of t shares does not leak any information about x_i
  - $t+1$ shares reveal x_i
- P_i sends to each P_j the value $f_i(j)$.
- The protocol continues from the input wires to the output wires.

Computation phase

- All parties participate in the computation of every gate
  - Already know a sharing of its input wires
  - Must generate a sharing of the output wire
- Addition gate: $c = a+b$
  - Must generate a polynomial $f_c()$ of degree t, which is random except for $f_c(0)=a+b$. Each P_i learns $f_c(i)$.
  - Define $f_c(\cdot) = f_a(\cdot)+f_b(\cdot)$
  - Each Pi sets $c_i=a_i+b_i=f_a(i)+f_b(i)=f_c(i)$
  - No interaction is needed!
- What about multiplication gates?

Output phase

- Easier to first describe the output phase than to describe the protocol for multiplication gates

- Output wires
  - If output wire y_i must be learned by P_i, then all parties send it their shares of y_i.
  - P_i reconstructs the secret and learns the output value.

Computation phase – multiplication gates

- $c = a \cdot b$. First attempt:
  - Define $f_{ab}(\cdot) = f_a(\cdot) \cdot f_b(\cdot)$.
  - Each P_i computes $a_i \cdot b_i = f_a(i) \cdot f_b(i) = f_{ab}(i)$.
  - Indeed, $f_{ab}(0) = a \cdot b$.
  - But the degree of f_{ab} is $2t$, and f_{ab} is not a random polynomial.
- Interpolation:
  - f_{ab} is of degree $2t<n$, and $f_{ab}(0) = a \cdot b$.
  - Therefore $\exists$ Lagrange coefficients $r_1,\ldots,r_n$ s.t.

    $f_{ab}(0) = a \cdot b = r_1 f_{ab}(1) + \ldots r_n f_{ab}(n) = r_1 \cdot a_1 b_1 + \ldots r_n \cdot a_n b_n$.
  - Each r_i is easily computable.

Computation phase – multiplication gates

- Each P_i
  - Has $a_i \cdot b_i$
  - Creates a random polynomial $g_i(\cdot)$ of degree t s.t. $g_i(0)=a_i \cdot b_i$

- Consider $g(x)=\Sigma_{i=1\ldots n} r_i \cdot g_i(x)$
  - of degree t
  - $g(0)=\Sigma_{i=1\ldots n} r_i \cdot g_i(0) = \Sigma_{i=1\ldots n} r_i \cdot a_i b_i = \Sigma_{i=1\ldots n} r_i \cdot f_{ab}(i) = a \cdot b$.
  - This is exactly the polynomial we need.
  - Must provide each P_i with a share of $g()$.

Computation phase – multiplication gates

- Each P_i
  - Creates a random polynomial $g_i(\cdot)$ of degree t s.t. $g_i(0)=a_i \cdot b_i$
  - Define $g(x)=\Sigma_{i=1\ldots n} r_i \cdot g_i(x)$, of degree t. $g(0)=\Sigma_{i=1\ldots n} r_i \cdot g_i(0) = a \cdot b$.

- P_i sends to every P_j the value $g_i(j)$
- Every P_j receives $g_1(j),\ldots,g_n(j)$, computes $g(j)= \Sigma_{i=1\ldots n} r_i \cdot g_i(j)$
- This is the desired sharing of $a \cdot b$.

Properties

- Correctness is straightforward

- Overhead:
  - $O(n^2)$ messages for every multiplication gate
  - # of rounds linear in depth of circuit (where only multiplication gates count)

Security

- Main idea: every set of t players, receives in each round values which are t-wise independent, and therefore uniformly distributed.
  - Therefore no information about the actual wire values are leaked.

Simulation based proof

- Recall what we showed
  - In (t+1)-out-of-n secret sharing, any t shares are uniformly distributed, independently of the secret.

- Suppose first that multiplication is computed by an oracle (call this the f_{mult} hybrid model)
  - The simulator obtains the inputs and outputs of the t corrupt parties
  - The transcript of a party includes its input, randomness used, all messages received.

Simulation based proof

- Adversary controls a set J of $t < n/2$ parties.
- The simulator:
  - $\forall P_i \in J$, set input $z_i = x_i$. $\forall\ P_i \notin J$, set input $z_i = 0$.
  - Share inputs z_i according to protocol.
  - Addition gates: add shares as in protocol.
  - Mult gates: provide $P_i \in J$ with shares of a random sharing of the value 0.
  - Simulation is correct since t shares of any value are uniformly distributed.

Simulation based proof

- Output stage:
  - $\forall$ wire, the simulator already defined shares for all $P_i \in J$.
  - Let w be an output wire of $P_i \in J$. The simulator has the output value y_w, and the t shares of $P_i \in J$.
  - The simulator interpolates the t-degree polynomial f_w going through these values. It then simulates receiving the shares $f_w(i)$ from all $P_i \notin J$.
  - Let w be an output wire of $P_j \notin J$. For all $P_i \in J$, the simulator sends the corresponding share to P_i.

Simulating the multiplication protocol

- Recall, the multiplication protocol
  - P_i creates a random poly $g_i(\cdot)$ of deg t s.t. $g_i(0)=a_i \cdot b_i$
  - P_i sends to $\forall$ P_j the value $g_i(j)$, and receive shares $g_j(i)$
  - P_i computes its share as $g(i)= \Sigma_{j=1\ldots n} r_j \cdot g_j(i)$.
  -
- Simulation $\forall$ $P_i \in J$:
  - Create a random poly $g_i(\cdot)$ of deg t s.t. $g_i(0)=P_i$'s share
  - Send to every P_j the value $g_i(j)$
  - $\forall P_j \notin J$ simulate receipt of a random share $g_j(i)$
  - Compute share of wire value as $g(i)= \Sigma_{j=1\ldots n} r_j \cdot g_j(i)$

Security against malicious parties

- Aka security against Byzantine adversaries
- Possible problems in using the previous protocol:
  - When sharing its input, P_i might send values of a polynomial of degree greater than t.
    - As a result, different subsets of the clients might recover different values as the secret.
  - Parties might send incorrect shares
    - How can we interpolate in this case?
- Protocol secure against t<n/3

Major tool – **Verifiable Secret Sharing** (VSS)

- Sharing stage
  - Add elements to the shares so that parties are assured to receive values of a polynomial of degree t (even if the dealer is malicious)
- Recovery stage
  - As long as $t<n/3$ shares are corrupt, use error correction techniques to recover the secret.
  - Based on the fact that Shamir's secret sharing scheme is a Reed-Solomon code, which can correct up to $t<n/3$ errors.

The Reed-Solomon code

- Reed-Solomon code
  - A linear [n,k,d]-code, with k=t+1, and d=n-t.
  - The message is $(m_0,\ldots m_t)$.
  - Use it as the coefficients of a degree t polynomial, P_m.

  - Codeword is $\langle P_m(1),\ldots,P_m(n)\rangle$.
  - Two codewords differ in at least d=n-t locations.
  - $\exists$ efficient decoding correcting (n-t-1)/2 errors.
  - If t<n/3, correcting up to t errors.

Using the Reed-Solomon code

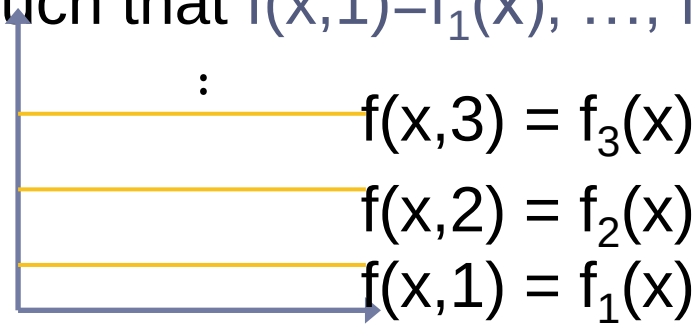
- Usage:
  - Let P() be a polynomial of degree t. (E.g., the polynomial used for (t+1)-out-of-n secret sharing.)
  - If instead of receiving ⟨P(1),P(2),…,P(n)⟩, we receive up to $t<n/3$ corrupt values, can still recover P.

    (And in particular, recover P(0), the secret.)

- Conclusion:
  - Can easily handle corrupt parties which send corrupt shares.
  - Need to focus on forcing the dealer to distribute shares consistent with a t-degree polynomial.

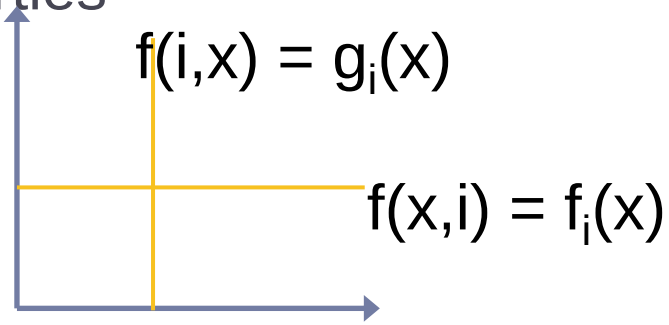
Bivariate polynomials

- $f(x,y) = \Sigma_{i=0\ldots t}\Sigma_{j=0\ldots t}\ a_{i,j}\cdot x^i\cdot y^j$
  - Defined by $(t+1)^2$ coefficients
  - Claim: $f(x,y)$ can be defined by $t+1$ univariate polynomials:
    - Given $t+1$ polynomials of degree t: $f_1(x),\ldots,f_{t+1}(x)$ there exists a single bivariate polynomial of degree t such that $f(x,1)=f_1(x)$, …, $f(x,t+1)=f_{t+1}(x)$



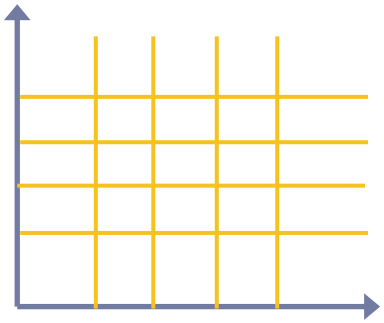
VSS using Bivariate polynomials - Step 1 (t+1)-out-of-n secret sharing

- Dealer defines a random bivariate polynomial f(x,y) of degree t, s.t. f(0,0)= secret.
- Sends to P_i the share $f_i(x)=f(x,i)$. (t-deg poly)
  - By the claim, any t+1 shares suffice to reveal secret.
- Sends to P_i the dual share $g_i(x)=f(i,x)$.
  - Will be used for checking shares received from other parties



VSS using Bivariate polynomials

- Claim: $\forall$subset J of size t, the shares and dual shares of $P_i \in J$ do not reveal the secret.
  - Assume wlog $J=1,2,\ldots,t$.
  - $f_1(x),\ldots,f_t(x)$, each of degree t, enforce $t\cdot(t+1)$ constraints of the bivariate polynomial f.
  - $g_1(x),\ldots,g_t(x)$, each add another constraint.
  - Total # of constraints is $t(t+1)+t=t^2+2t=(t+1)^2-1$. None of them defines $f(0,0)$ directly.

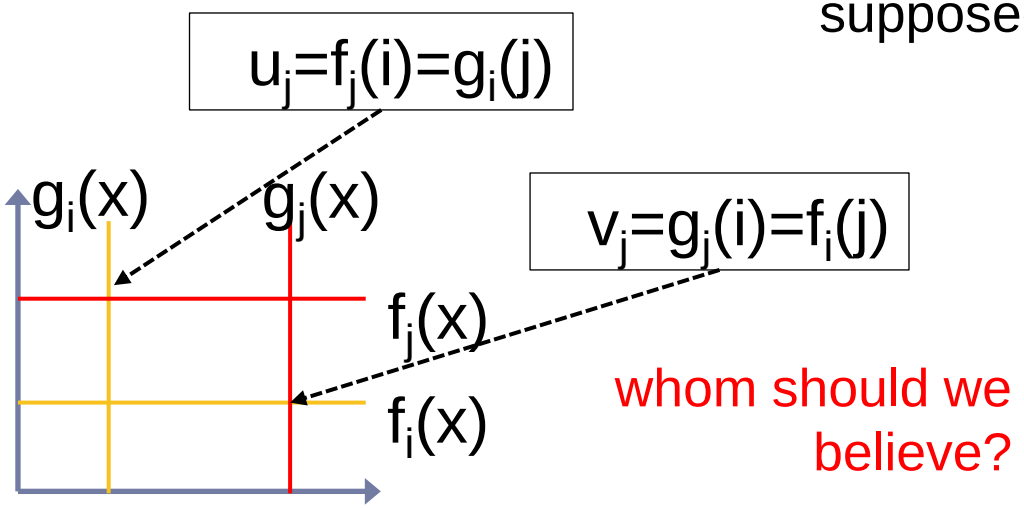


VSS using Bivariate polynomials – Step 2

- Each party P_i:
  - $\forall$ j, send $f_i(j)$ and $g_i(j)$ to P_j.
  - $\forall$ j, let (u_j,v_j) the values received from P_j.

    If $u_j \neq g_i(j)$ or $v_j \neq f_i(j)$, then broadcast "complaint(i, j, $f_i(j)$, $g_i(j)$)".

(the two values P_i was supposed to receive)



VSS using Bivariate polynomials – Step 3

- The dealer:
  - Upon receiving the message "complaint(i, j, $f_i(j)$, $g_i(j)$)" sent by P_i,

    check that $f_i(j)=f(i,j)$ and that $g_i(j)=f(j,i)$.
  - If the checks fail, broadcast polynomials: reveal(i,$f_i(x)$,$g_i(x)$).
  - (Namely, if P_i sent an incorrect complaint, broadcast the shares that it received from dealer.)

- Now, whom should the parties believe, P_i or the dealer?

VSS using Bivariate polynomials – Step 4

- Each P_i
  1. If P_i views two messages complaint(k,j,u_1,v_1) and complaint(j,k,u_2,v_2), and the dealer did not broadcast a corresponding reveal message, go to 3.

  2. If P_i views a message reveal(j,$f_j(x)$,$g_j(y)$), check if it agrees with P_i's shares: $f_i(j)=g_j(i)$ and $g_i(j)=f_j(i)$. If the check succeeds, broadcast "good" (i.e., I agree with the dealer).

  3. If at least n-t parties broadcasted "good" then use the shares that they have. Otherwise they abort.

VSS Security proof - Sketch

- Assume dealer is honest
  - An honest P_J complains only if a corrupt P_i sends it incorrect values. But since the complaint of P_i contains good values, the dealer does not reveal P_J's share.
  - If a corrupt P_i complains with incorrect values, dealer sends a reveal message of P_i's shares,

    which passes the test of the n-t honest parties,

    which then send n-t good messages

    and therefore output the correct shares which enable to recover the secret.

VSS Security proof - Sketch

- Assume dealer is corrupt
  - Suppose P_i,P_k are honest and receive inconsistent shares: $f_j(k) \neq g_k(j)$, or $g_j(k) \neq f_k(j)$.
  - Both parties complain, and therefore dealer must send reveal message or else no honest party broadcasts good.
  - The shares are used only if n-t parties output “good”. Some might be corrupt, but at least (n-t)-t=t+1 of them are honest.
  - Their polynomials agree with those revealed by the dealer.
  - These t+1 polynomials define a unique bivariate poly, which defines the secret.
- That’s all that we need.

The full protocol

- Inputs are shared using VSS.
  - Therefore dealer deals consistent shares.
- Addition gates are trivial.
- Multiplication gates:
  - Must ensure that each party multiplies its own shares.
  - Must use a VSS to perform the sharing defined by the protocol.
  - The full description and proof are quite intricate.

Overhead

- No public key operations are needed!
- Input sharing step is more complicated than in the semi-honest case
  - Length of messages increases by O(n)
  - But this protocol is run only once, and has O(1) rounds.
- Multiplication gates
  - Requires the use of a VSS
  - Message length increases by O(n)