Advanced Topics in Cryptography

Lecture 9
Secure Two-Party and Multi-Party Computation

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- In the last class we learned Yao's protocol for secure two-party computation
- The protocol is based on first representing the function as a Boolean circuit

Example application

- Comparing two N bit numbers

- What's the overhead?

Applications

- Two parties. Two large data sets.
- Max?
- Mean?
- Median?
- Intersection?
- Sorting? (useful as a subcircuit)

Conclusions

- If the circuit is not too large:
  - Efficient secure two-party computation.
  - Also, efficient multi-party computation with two semi-trusted parties.
    - Many parties with private inputs
    - Two designated parties that are assumed not to collude
    - Each party with input x_i sends the two parties random shares x_i^1, x_i^2 such that $x_i^1 \oplus x_i^2 = x_i$.
    - The two designated parties run the computation.
- If the circuit is large: we currently need ad-hoc solutions.

A two-party protocol for a function which does not have a short circuit

Related papers

- Secure computation of medians
  - G. Aggarwal, N. Mishra and B. Pinkas, *Secure Computation of the K'th-ranked Element,* Eurocrypt '2004.

Secure Function Evaluation

- Yao’s protocol is efficient for medium size circuits, but what about functions that cannot be represented as small circuits?

k^{th}-ranked element (e.g. median)

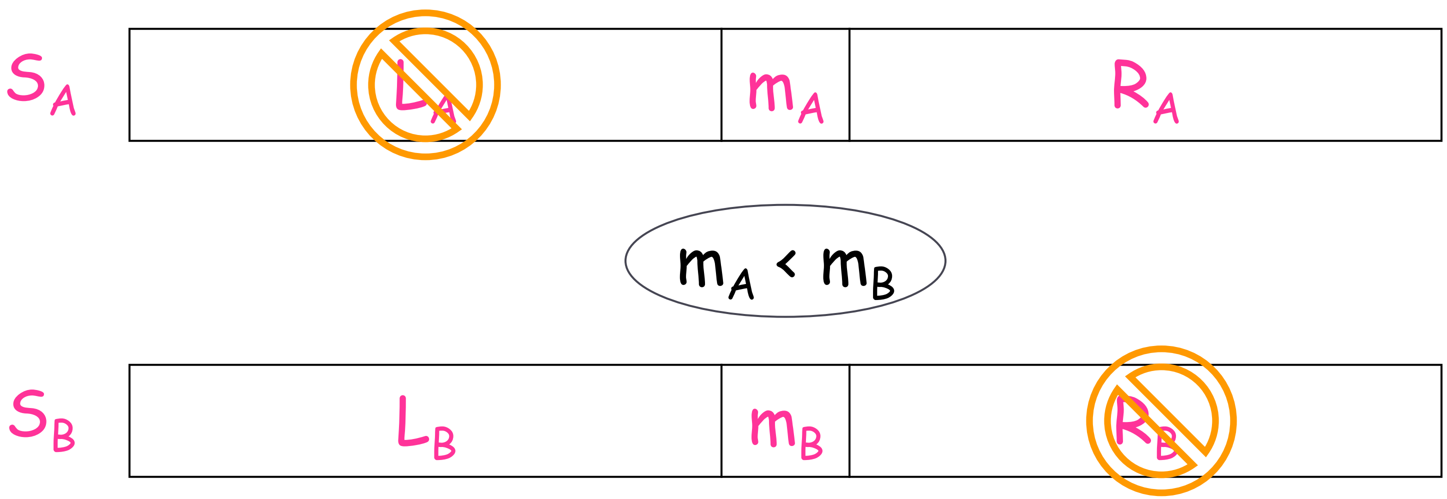
- Inputs:
  - Alice: S_A Bob: S_B
  - *Large* sets of unique items ($\in$D).
- Output:
  - $x \in S_A \cup S_B$ s.t. x has k-1 elements smaller than it.
- The rank k
  - Could depend on the size of input datasets.
  - Median: $k = (|S_A| + |S_B|) / 2$
- Motivation:
  - Basic statistical analysis of distributed data.
  - E.g. histogram of salaries in different comapnies

Secure computation in the case of large circuit representation

- The Problem:
  - The size of a circuit for computing the k^{th} ranked element is at least linear in k. This value might be very large.
  - Generic constructions using circuits [Yao …] have communication complexity which is linear in the circuit size, and therefore in k.

- It is sometimes possible to design specific protocols for specific problems, and obtain a much better overhead.
- We will show such a protocol for computing the k^{th} ranked element, for the case of semi-honest parties.

An (insecure) two-party median protocol

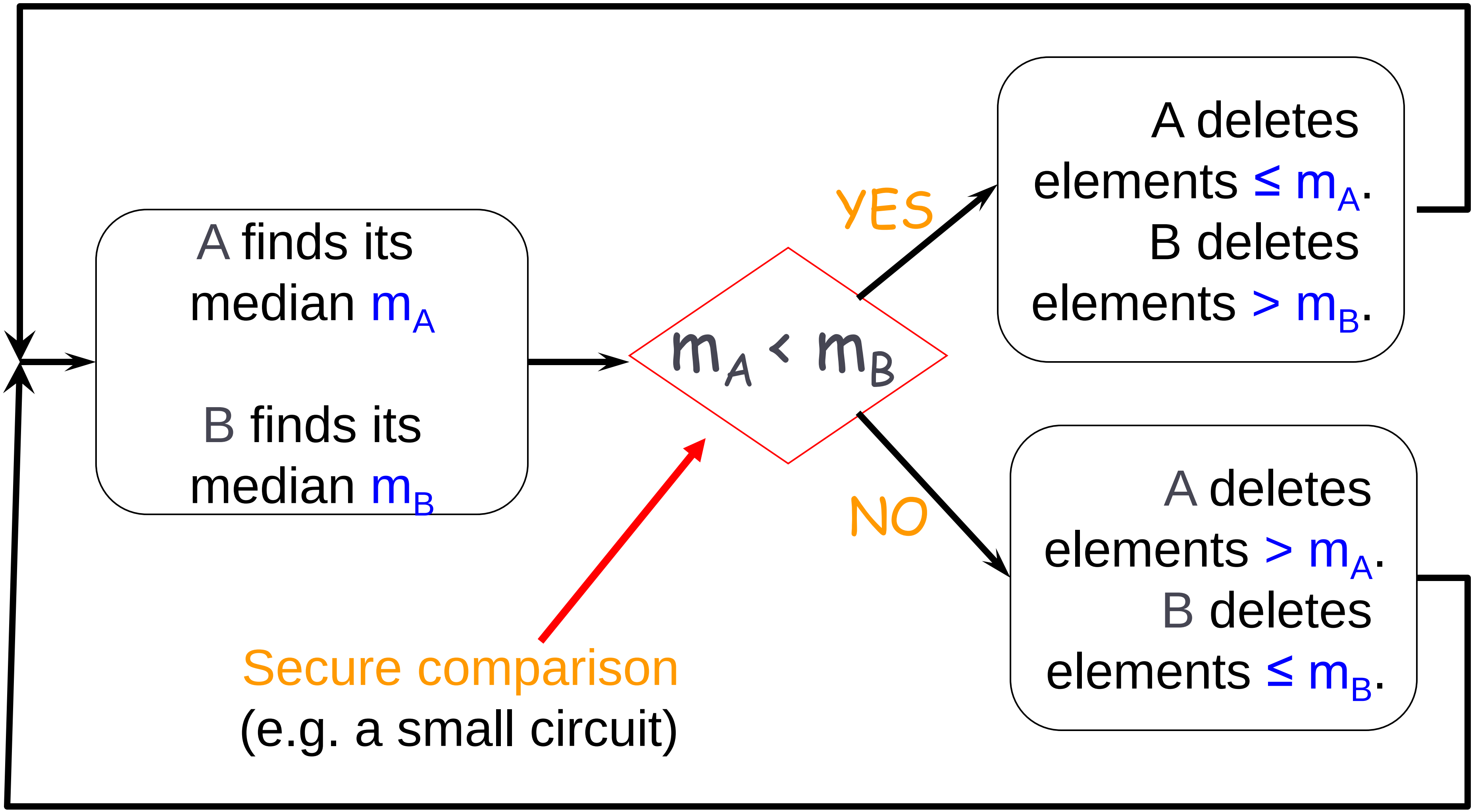


L_A lies below the median, R_B lies above the median. $|L_A| = |R_B|$

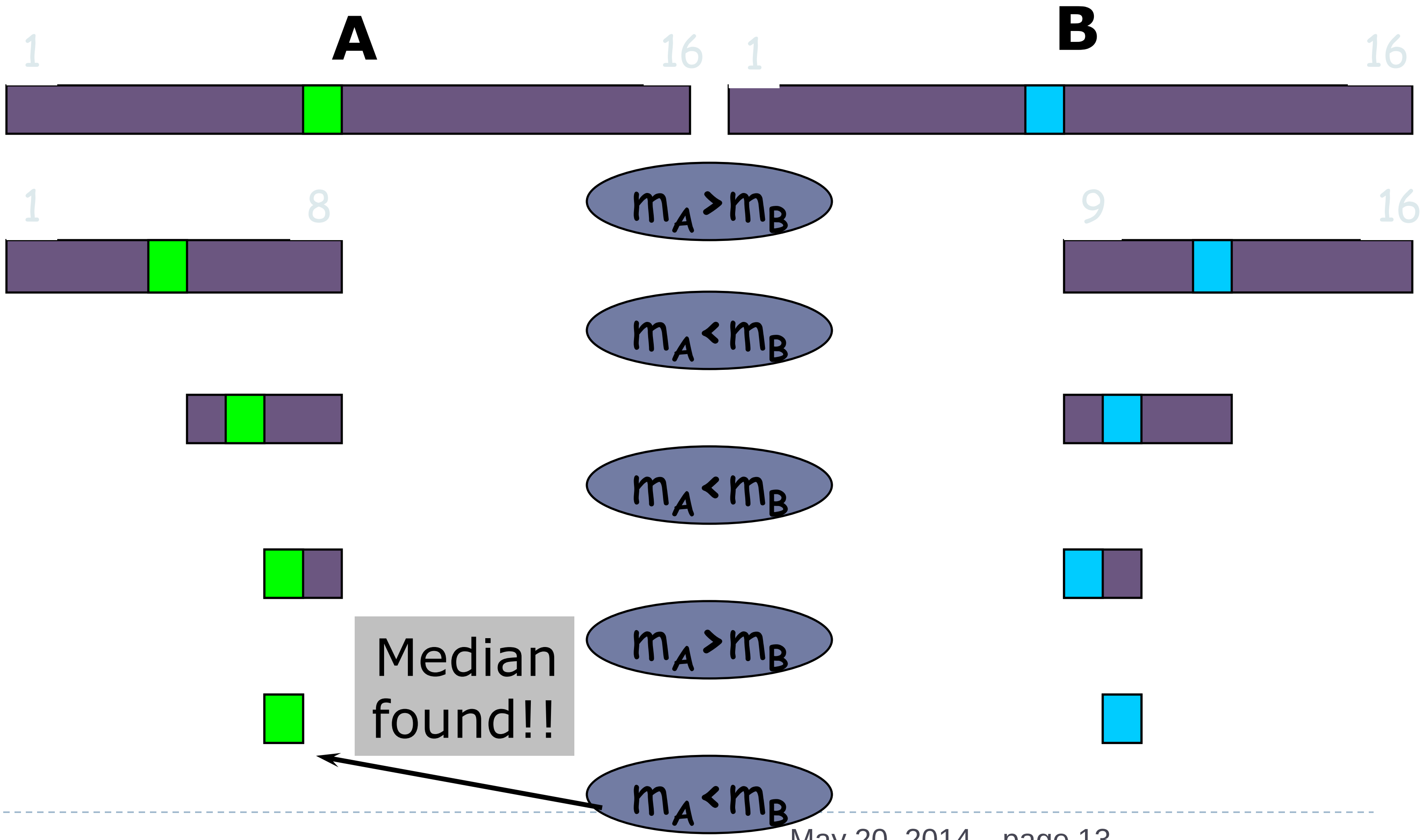
New median is same as original median.

Recursion → Need log n rounds
(assume each set contains $n=2^i$ items)

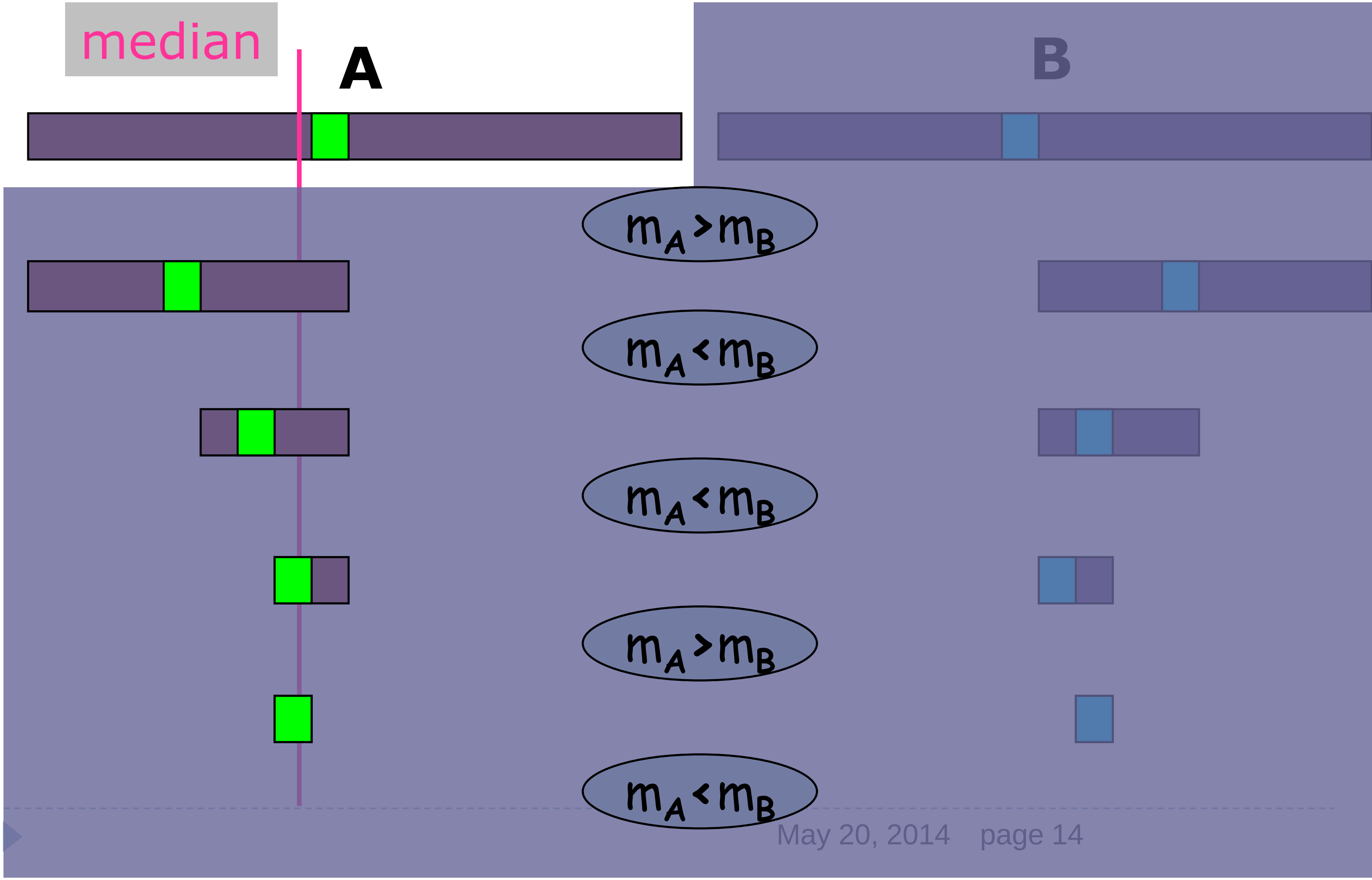
A Secure two-party median protocol



An example



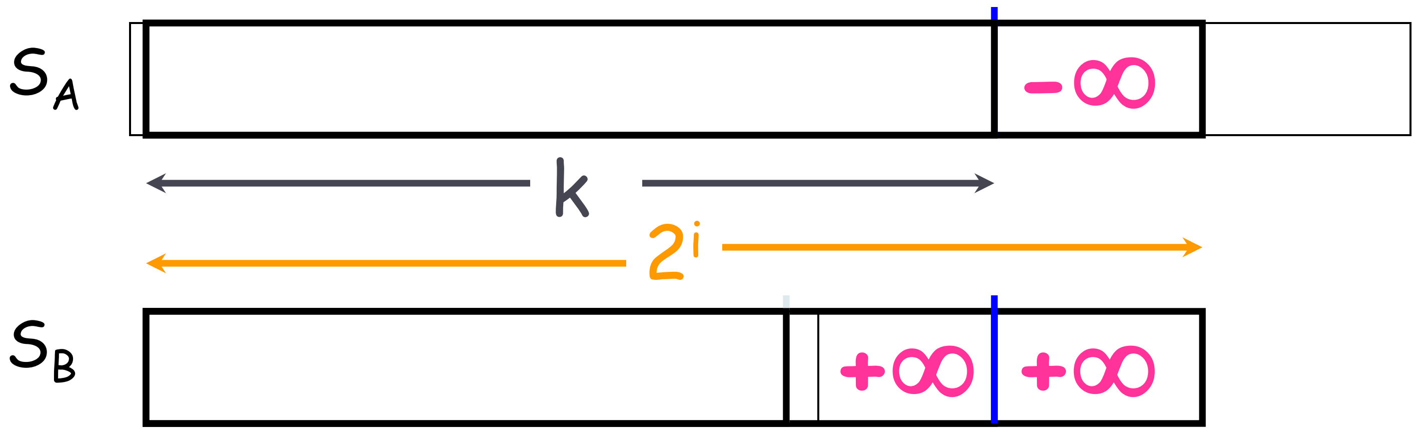
Proof of security



Proof of security

- This is a proof of security for the case of semi-honest adversaries.
- Security for malicious adversaries is more complex.
  - The protocol must be changed to ensure that the parties' answers are consistent with some input.
  - Also, the comparison of the medians must be done by a protocol secure against malicious adversaries.

Arbitrary input size, arbitrary k



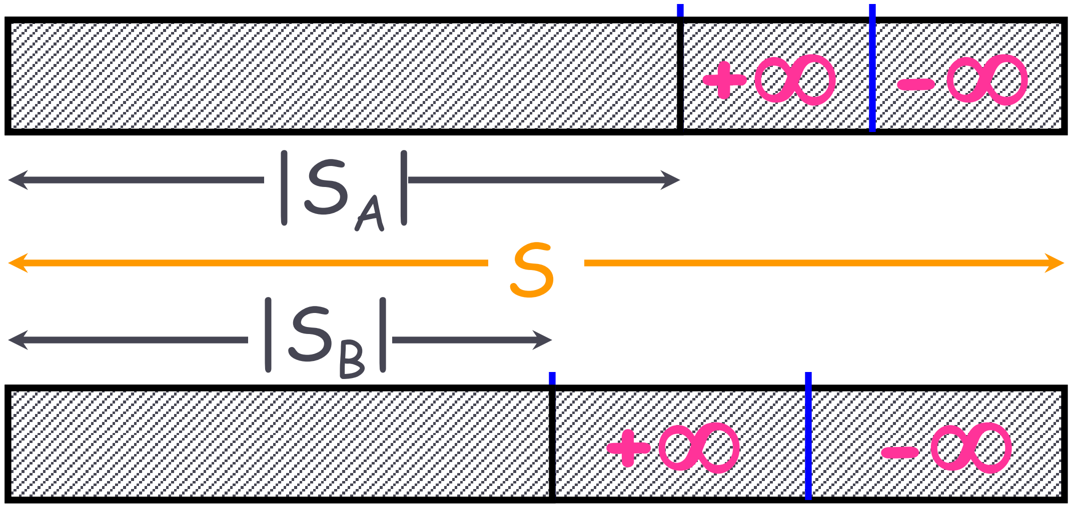
Now, compute the median of two sets of size k.

Size should be a power of 2.

$\boxed{\text{median of new inputs} = k^{th} \text{ element of original inputs}}$

Hiding size of inputs

- Can search for k^{th} element without revealing size of input sets.
- However, k=n/2 (median) reveals input size.
- Solution: Let $S=2^i$ be a bound on input size.



Median of new datasets is same as median of original datasets.

Secure Computation in the Multi-Party Setting

Overview

- Secure computation for more than two parties, computing Boolean circuits.

- GMW (Goldreich-Micali-Wigderson)
  - First, for semi-honest adversaries.
  - Then, general compiler from semi-honest to malicious
  - # rounds depends on circuit depth
  - O. Goldreich, Foundations of Cryptography, Vol. II, Chapter 7.
- BMR (Beaver-Micali-Rogaway)
  - O(1) rounds

The setting

- Parties $P_1,\ldots,P_n$
- Inputs $x_1,\ldots,x_n$ (bits, but can be easily generalized)
- Outputs $y_1,\ldots,y_n$

- The functionality is described as a Boolean circuit.
  - Wlog, uses only XOR (+) and AND gates
  - NOT(x) is computed as a x+1
  - Wires are **ordered** so that if wire k is a function of wires i and j, then i<k and j<k.

The setting

- The adversary controls a subset of the parties
  - This subset is defined before the protocol begins (is “non-adaptive”)
  - We will not cover the adaptive case

- Communication
  - Synchronous
  - Private channels between any pair of parties
    (can be easily implemented using encryption)

Adversarial models

- Semi-honest

- Malicious with no abort
  - GMW: A protocol secure any number of malicious parties

- Malicious with abort
  - GMW: A protocol secure against a minority of malicious parties with abort (will not be discussed here).

Protocol for semi-honest setting

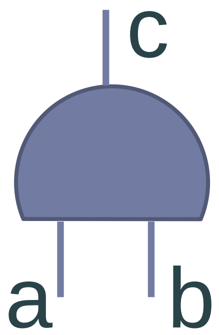
- The protocol:
  - Each party shares its input bit
  - Scan the circuit gate by gate
    - Input values of gate are shared by the parties
    - Run a protocol computing a sharing of the output value of the gate
    - Repeat
  - Publish outputs

Protocol for semi-honest setting

- The protocol:
  - Each party shares its input bit
  - The sharing procedure:
    - P_i has input bit x_i
    - It chooses random bits $r_{i,j}$ for all $i \neq j$.
    - Sends bit $r_{i,j}$ to P_j.
    - Sets its own share to $r_{i,i} = x_i + (\Sigma_{j \neq i} r_{i,j}) \bmod 2$
    - Therefore $\Sigma_{j=1\ldots n} r_{i,j} = x_i \bmod 2$.
  - Now every P_j has n shares, one for each input x_i of each P_i.

Evaluating the circuit

- Scan circuit by the order of wires
- Wire c is a function of wires a,b
  - P_i has shares a_i, b_i. Must get share of c_i .



  - Addition gate:
    - P_i computes $c_i=a_i+b_i$ mod 2.

  - Indeed, $c = a+b \pmod 2) =$
    $(a_1+\ldots+a_n) + (b_1+\ldots+b_n) = (a_1+b_1)+\ldots+(a_n+b_n) =$
    $c_1+\ldots+c_n$

Evaluating multiplication gates

- $c = a\cdot b = (a_1+\ldots+a_n) \cdot (b_1+\ldots+b_n) = \Sigma_{i=1\ldots n}\, a_ib_i + \Sigma_{i\neq j}\, a_ib_j = \Sigma_{i=1\ldots n}\, a_ib_i + \Sigma_{1\leq i<j\leq n}\, (a_ib_j + a_jb_i)$
- P_i will receive a share of $a_ib_i+\Sigma_{i<j\leq n}\, (a_ib_j + a_jb_i)$
- Computing a_ib_i by P_i is easy
- What about $a_ib_j + a_jb_i$?
- P_i and P_j run the following protocol for every $i<j$.

Evaluating multiplication gates

- Input: P_i has a_i, b_i, P_j has a_j, b_j.
- P_i outputs $a_i b_j + a_j b_i + s_{i,j}$. P_j outputs $s_{i,j}$.
- P_j:
  - Chooses a random $s_{i,j}$
  - Computes the four possible outcomes of $a_i b_j + a_j b_i + s_{i,j}$, depending on the four options for P_i's inputs.
  - Sets these values to be its input to a 1-out-of-4 OT

- P_i is the receiver, with input $2a_i + b_i$.

Recovering the output bits

- The protocol computes shares of the output wires.
- Each party sends its share of an output wire to the party P_i that should learn that output.
- P_i can then sum the shares, obtain the value and output it.

Proof of Security

- Recall definition of security for semi-honest setting:
  - Simulation - Given input and output, can generate the adversary's view of a protocol execution.

- Suppose that adversary controls the set J of all parties but P_i.
- The simulator is given (x_j,y_j) for all $P_j \in J$.

The simulator

- Shares of input wires: $\forall j \in J$ choose
  - a random share $r_{j,i}$ to be sent from P_j to P_i,
  - and a random share $r_{i,j}$ to be sent from P_i to P_j.
- Shares of multiplication gate wires:
  - $\forall j<i$, choose a random bit as the value learned in the 1-out-of-4 OT.
  - $\forall j>i$, choose a random $s_{i,j}$, and set the four inputs of the OT with P_i accordingly.

- Output wire y_j of $j \in J$: set the message received from P_i as the XOR of y_j and the shares of that wire held by $P_j \in J$.

Security proof

- The output of the simulation is distributed identically to the view in the real protocol
  - Certainly true for the random shares $r_{i,j}$, $r_{j,i}$ sent from and to P_i.
  - OT for $j<i$: output is random, as in the real protocol.
  - OT for $j<i$: input to the OT defined as in the real protocol.
  - Output wires: message from P_i distributed as in the real protocol.
- QED

Performance

- Must run an OT for every multiplication gate
  - Namely, public key operations per multiplication gate
  - Need a communication round between all parties per every multiplication gate

  - Can process together a set of multiplication gates if all their input wires are already shared
  - Therefore number of rounds is O(d), where d is the depth of the circuit (counting only multiplication gates).

The BMR protocol

- Beaver-Micali-Rogaway
- A multi-party version of Yao's protocol
- Works in O(1) communication rounds, regardless of the depth of the Boolean circuit.

  - D. Beaver, S. Micali and P. Rogaway, "The round complexity of secure protocols", 1990.
  - A. Ben-David, N. Nisan and B. Pinkas, "FairplayMP – A System for Secure Multi-Party Computation", 2010.

The BMR protocol

- Two random seeds (garbled values) are set for every wire of the Boolean circuit:
  - Each seed is a concatenation of seeds generated by all players and secretly shared among them.
- The parties securely compute together a 4x1 table for every gate (in parallel):
  - Given 0/1 seeds of the input wires, the table reveals the seed of the resulting value of the output wire.

The BMR protocol

- The parties securely compute together a 4x1 table for every gate (in parallel):
  - This is essentially a secure computation of the table
  - But all tables can be computed in parallel. Therefore O(1) rounds.
  - This is the main bottleneck of the BMR protocol.

- Given the tables, and seeds of the input values, it is easy to compute the circuit output.

The malicious case

- What can go wrong with malicious behavior?
  - Using shares other than those defined by the protocol, using arbitrary inputs to the OT protocol and sending wrong shares of output wires…
- We will show a compiler which forces the parties to operate as in the semi-honest model. (For both GMW and BMR.)
- The basic idea:
  - In every step, each P_i proves in zero knowledge that its messages were computed according to the protocol

Zero knowledge proofs
(we studied this already)

- Prover P, verifier V, language L
- P proves that $x \in L$ without revealing anything
  - Completeness: V always accepts when $x \in L$, and an honest P and V interact.
  - Soundness: V accepts with negligible probability when $x \notin L$, for any P^*.
    - Computational soundness: only holds when P^* is polynomial-time
- Zero-knowledge:
  - There exists a simulator S such that S(x) is indistinguishable from a real proof execution.

A warm-up

- Assume that each party P_i runs a deterministic program Π_i. The compiler is the following:
  - Each P_i commits to its input x_i by sending $C_i(r_i,x_i)$, where r_i is a random string used for the commitment.
  - Let T_i^s be the transcript of P_i at step s, i.e. all messages received and sent by P_i until that step.
  - Define the language $L_i = \{T_i^s$ s.t. $\exists x_i,r_i$ so that all messages sent by P_i until step s are the output of Π_i applied to x_i,r_i and to all messages received by P_i up to that step$\}$
  - When sending a message in step s prove in zero-knowledge that $T_i^s \in L_i$.

Handling randomized protocols

- The previous construction assumes that P_i's program, Π_i, is deterministic.

- This is not true in the semi-honest protocol we have seen.
  - In particular, the choice of shares, and the sender's input to the OT, must be random.
  - The compiler must ensure that P_i chooses its random coins independently of the messages received from other parties.
  - This is not ensured by the previous construction.

The compiler

- We will describe the basic issues of a protocol secure against any number of malicious parties, but with no aborts allowed.
- Communication model:
  - Messages are published on a bulletin board, and can be read by all parties.
  - This implements a broadcast, ensuring that all parties receive the same message.
  - Broadcast can be easily implemented if a public key infrastructure exists.
  - We assume that a PKI does exist.

The compiler

- Input commitment phase:
  - Each party commits to its input.
- Coin generation phase:
  - The parties generate random tapes for each other (this ensures that the randomness is independent of the messages.)
  - Initial idea: random tape of P_i is defined as $s_{1,i} \oplus s_{2,i} \oplus \ldots \oplus s_{n,i}$, where $s_{j,i}$ is chosen by P_j.
  - But this lets P_n control the outcome ☹
- Protocol emulation phase:
  - Run the protocol while proving that the operations of the parties comply with their inputs and random tapes.

The protocol: Input commitment phase

- The required functionality for P_1 is $(x,1^{|x|},\ldots 1^{|x|}) \rightarrow (r,C_r(x),\ldots C_r(x))$, and similarly for each P_i.
  - (This is required in order to choose the randomness.)
- It is not sufficient to ask P_1 to just broadcast a commitment of its input
  - This does not ensure that this is a random commitment for which P_i knows a decommitment.
- The protocol is more complex…
- It is useful to first design tools that can help in constructing the compiler.

Tool 1: image transmission

- The required functionality is $(a,1^{|a|},\ldots 1^{|a|})\rightarrow(\lambda,f(a),\ldots,f(a))$ (<u>all</u> receive the same function of a)
- Protocol
  - P_1 broadcasts an encryption of f(a) (f() is a public function)
  - For j=2…n, P_1 proves to P_j a zero-knowledge proof of knowledge of a value a corresponding to f(a).
  - If P_j rejects, it broadcasts the coins it used in the proof.
- Output: For j=2…n, if P_j sees a justifiable rejection it aborts, otherwise it outputs f(a).
  - Agreement to whether P_1 misbehaved is reduced to the decision on whether some verifier has justifiably rejected the proof.

Tool 1: image transmission

- The required functionality is $(a,1^{|a|},\ldots 1^{|a|}) \longrightarrow (\lambda, f(a), \ldots, f(a))$
- Agreement as to whether P_1 misbehaved is reduced to the decision on whether some verifier has justifiably rejected the proof.
- If P_1 is honest, then no malicious party can claim that it cheated.

Tool 2: authenticated computation

- The required functionality is
  $(a,b_2,\ldots,b_n) \rightarrow (\lambda,v_2, \ldots,v_n)$,
  where $v_j=f(a)$ if $b_j=h(a)$ and $v_j=\lambda$ otherwise.
- Protocol:
  - Use the image transmission tool to broadcast $(f(a),h(a))$ to all P_j, $j=2\ldots n$.
  - P_j outputs $f(a)$ if $b_j=h(a)$, and λ otherwise.

- Comment: P_j learns a function $f(a)$ of a, if it already has the function $h(a)$ (e.g., if it has a commitment to a)

Tool 3: multi-party augmented coin-tossing

- The required functionality is

  $(1^n,,\ldots,1^n)\rightarrow(r,g(r),\ldots, g(r))$.

- Typically we will use it for computing $(1^n,,\ldots,1^n)\rightarrow((r,s), C_s(r),\ldots, C_s(r))$, where r is random.

- The challenge: ensuring that P_1's output is random. We cannot trust P_1 to choose a random output.

Tool 3: multi-party augmented coin-tossing

- $(1^n,,\ldots,1^n) \to ((r,s), C_s(r),\ldots, C_s(r))$.
  - **Toss** and commit: $\forall i$, P_i chooses r_i,s_i and uses the image transmission tool to send $c_i=C_{Si}(r_i)$ to all P_j.
  - **Open** commits: $\forall i\geq 2$, P_i uses the authenticated computation tool to send s_i,r_i to all parties that already have c_i.
  - If P_j obtains r_i agreeing with c_i, it sets $r_i^j=r_i$ (also, $r_j^j=r_j$).

  Otherwise it aborts.
  - If P_1 did not abort, it sets $r=\bigoplus_{i=1\ldots n} r_i$, sends $C_s(r)$ to all other parties (to be used for the main protocol), and proves that $C_s(r)$ was constructed correctly.
    - (details in the next slide)

Tool 3: multi-party augmented coin-tossing (contd.)

- P_1 sends $C_s(r)$ to all other parties, and proves that it was constructed correctly.

- Run the authenticated computation functionality
  - P_1 chooses a random s. Its input to the protocol is $(r_1,s_1,s,\oplus_{j=2\ldots n}r_i^1)$
  - P_j's input is c_1, $\oplus_{j=2\ldots n}r_i^j$.
  - If $c_1=C_{S1}(r_1)$ and $\oplus_{j=2\ldots n}r_i^j = \oplus_{j=2\ldots n}r_i^1$, then P_j outputs $C_s(\oplus_{j=1\ldots n}r_i) = C_s(r)$. Otherwise it aborts.
  - P_1 outputs r.

The main protocol:
Input commitment phase

- Protocol:
  - P_i chooses random r'_i and uses the image transmission functionality to send $c'=C_{r'_i}(x_i)$ to all parties.
  - Run augmented coin-tossing protocol s.t. P_i learns (r_i,r''_i) and others learn $c''=C_{r''_i}(r_i)$.
  - Run authenticated computation where P_i has input (x_i,r_i,r'_i,r''_i) and others input (c',c''), and others learn $C_{ri}(x_i)$ if (c',c'') are the required functions of P_i's input.

The main protocol:
coin generation phase

- Each P_i runs the augmented coin tossing protocol where
  - P_i learns (r^i, s^i)
  - The other parties learn $C_{si}(r^i)$.

The main protocol: Protocol emulation phase

- The parties use the authenticated computation functionality
  - $(a,b_2,\ldots,b_n)\rightarrow(\lambda,v_2,\ldots,v_n)$, where $v_j=f(a)$ if $b_j=h(a)$ and $v_j=\lambda$ otherwise.
- Suppose that it is P_i's turn to send a message
  - Its input is (x_i,r^i,T_t), as well as the coins used for commitments, where T_t is the sequence of messages exchanged so far.
  - Every other party has input $(C(x_i),C(r^i),T_t)$
  - $f(x_i,r^i,T_t)$ is the message P_i must send
  - It is accepted if $(C(x_i),C(r_i),T)$ agree with x_i,r_i,T and the program that is run

Summary

- Can compute any functionality securely in presence of semi-honest adversaries
- Protocol is efficient enough for use, for circuits that are not too large
- The full proof is in Goldreich's book.