Introduction to Cryptography
Lecture 6

Diffie-Hellman Key Exchange

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Groups we will use

- Z_p^* Multiplication modulo a prime number p
  - $(G, \circ) = (\{1,2,\ldots,p\text{-}1\}, \times)$
  - E.g., $Z_7^* = (\{1,2,3,4,5,6\} , \times)$

- Z_N^* Multiplication modulo a composite number N
  - $(G, \circ) = (\{a \text{ s.t. } 1 \leq a \leq N\text{-}1 \text{ and } gcd(a,N)=1\}, \times)$
  - E.g., $Z_{10}^* = (\{1,3,7,9\}, \times)$

- A group G is cyclic if there exists a generator g, s.t. $\forall a \in G, \exists i$ s.t. $g^i = a$.
  - I.e., $G = <g> = \{1, g, g^2, g^3, \ldots\}$
  - For example $Z_7^* = <3> = \{1,3,2,6,4,5\}$

Number Theory

- Lagrange's Theorem: $\forall a$ in a finite group G, $a^{|G|}=1$.
- Euler's phi function (aka, Euiler's totient function),
  - $\varphi(n) =$ number of elements in Z_n^* (i.e. $| \{x \mid gcd(x,n)=1, 1 \leq x \leq n\} |$
  - $\varphi(p) = p\text{-}1$ for a prime p.
  - $n = \prod_{i=1..k} p_i^{e(i)} \Rightarrow \varphi(n) = n \cdot \prod_{i=1..k} (1\text{-}1/p_i)$
  - $\varphi(p^2) = p(p\text{-}1)$ for a prime p.
  - $n = p \cdot q \Rightarrow \varphi(n) = (p\text{-}1)(q\text{-}1)$

- Corollary: $\forall a \in Z_n^*$ it holds that $a^{\varphi(n)} = 1$ mod n
  - For Z_p^* (prime p), $a^{p\text{-}1} = 1$ mod p (Fermat's theorem).
  - For Z_n^* ($n = p \cdot q$), $a^{(p\text{-}1)(q\text{-}1)} = 1$ mod n

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Finding prime numbers

- Prime number theorem: #{primes $\leq x$} $\approx x / \ln x$ as $x \to \infty$
- How can we find a random k-bit prime?
  - Choose x at random in $\{2^k,\ldots,2^{k+1}\text{-}1\}$
  - Test if x is prime
- The probability of success is $\approx 1/\ln(2^k) = O(1/k)$.
- The expected number of trials is $O(k)$.

Finding generators

- How can we find a generator of Z_p^*?
- Can check whether $\forall \, 1 \le i \le p-2 \quad a^i \ne 1$ ☹
- We know that if $a^i = 1 \bmod p$ then $i \mid p-1$.
- Therefore need to check only i for which $i \mid p-1$.

- Easy if we know the factorization of $(p-1)$
  - For all $a \in Z_p^*$, the order of a divides $(p-1)$
  - For every integer divisor b of $(p-1)$, check if $a^b = 1 \bmod p$.
  - If none of these checks succeeds, then a is a generator.
  - a is a generator iff $ord(a) = p-1$.

Finding prime numbers of the right form

- How can we know the factorization of p-1
- Easy, for example, if $p=2q+1$, and q is prime.
- How can we find a k-bit prime of this form?

1. Search for a prime number q of length $k-1$ bits. (Will be successful after about $O(k)$ attempts.)
2. Check if $2q+1$ is prime.
3. If not, go to step 1.

Quadratic Residues

- The square root of $x \in Z_p^*$ is $y \in Z_p^*$ s.t. $y^2 = x \bmod p$.
- Examples: sqrt(2) mod 7 = 3, sqrt(3) mod 7 doesn't exist.
- How many square roots does $x \in Z_p^*$ have?
  - If a and b are square roots of x, then $x = a^2 = b^2 \bmod p$. Therefore $(a-b)(a+b) = 0 \bmod p$. Therefore either $a=b$ or $a=-b$ modulo p.
  - Therefore x has either 2 or 0 square roots, and is denoted as a Quadratic Residue (QR) or Non Quadratic Residue (NQR), respectively.
- $a^{(p-1)/2}$ is either 1 or -1 in Z_p^*. (indeed, $(a^{(p-1)/2})^2$ is always 1)
- Euler's theorem: $x \in Z_p^*$ is a QR iff $x^{(p-1)/2} = 1 \bmod p$.
- *Legendre's symbol:*
$$\left(\frac{x}{p}\right) = \begin{cases} 1 & x \text{ is a QR in } Z_p^* \\ -1 & x \text{ is an NQR in } Z_p^* \\ 0 & x = 0 \bmod p \end{cases}$$
- Can be efficiently computed as $x^{(p-1)/2} \bmod p$.

Hard problems in cyclic groups

- The following problems are believed to be hard in Z_p^* or in some subgroups of Z_p^*
  - Discrete logarithm: let g be a generator of G. The input is a random $x \in G$. The task is to find an r s.t. $x = g^r \bmod p$.

  - The Diffie-Hellman problem: The input contains g and random $x,y \in G$, such that $x = g^a$ and $y = g^b$. The task is to find $z = g^{a \cdot b}$.

  - The Decisional Diffie-Hellman problem: The input contains random $x,y \in G$, such that $x = g^a$ and $y = g^b$; and a pair (z,z') where one of (z,z') is $g^{a \cdot b}$ and the other is g^c (for a random c). The task is to tell which of (z,z') is $g^{a \cdot b}$.

- Solving DDH < solving DH < solving DL

Does the DDH assumption hold in Z_p^*?

- The DDH assumption does not hold in Z_p^*
  - Assume that both $x=g^a$ and $y=g^b$ are QRs in Z_p^*.
  - Namely, their Legendre symbol is 1, both a and b are even, and it holds that $x^{(p-1)/2}= y^{(p-1)/2}=1$.
  - Then the Legendre symbol of g^{ab} is always 1, whereas the symbol of a random g^c is 1 with probability $\frac{1}{2}$.
- Solution: (work in a subgroup of prime order)
  - Set $p=2q+1$, where q is prime.
  - $\varphi(Z_p^*) = p-1 = 2q$. Therefore Z_p^* has a subgroup H of *prime* order q.
  - Let g be a generator of H.
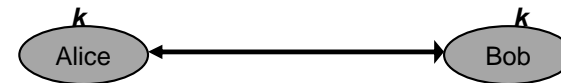 - The DDH assumption is believed to hold in H. (The Legendre symbol is always 1.)

Classical symmetric ciphers

- Alice and Bob share a private key k.
- System is secure as long as k is secret.
- Major problem: generating and distributing k.

Diffie and Hellman: "New Directions in Cryptography", 1976.

- "We stand today on the brink of a revolution in cryptography. The development of cheap digital hardware has freed it from the design limitations of mechanical computing…

  …such applications create a need for new types of cryptographic systems which minimize the necessity of secure key distribution…

  …theoretical developments in information theory and computer science show promise of providing provably secure cryptosystems, changing this ancient art into a science."

Diffie-Hellman

- Came up with the idea of public key cryptography

  public key$_{Bob}$ *secret key$_{Bob}$*

  Everyone can learn Bob's public key and encrypt messages to Bob. Only Bob knows the decryption key and can decrypt.

  Key distribution is greatly simplified.

- Diffie and Hellman did not have an implementation for a public key encryption system
- Suggested a method for key exchange over insecure communication lines, that is still in use today.

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Public Key-Exchange

- Goal: Two parties who do not share any secret information, perform a protocol and derive the same shared key.

- No eavesdropper can obtain the new shared key (if it has limited computational resources).

- The parties can therefore safely use the key as an encryption key.

The Diffie-Hellman Key Exchange Protocol

- Public parameters: a group Z_p^* (where $|p|= 768$ or 1024, $p=2q+1$), and a generator g of $H \subset Z_p^*$ of order q.

- Alice:
  - picks a random $a \in [1,q]$.
  - Sends g^a mod p to Bob.

- Bob:
  - picks a random $b \in [1,q]$.
  - Sends g^b mod p to Bob.

  - *Computes $k=(g^b)^a$ mod p*
  - *Computes $k=(g^a)^b$ mod p*

- $K = g^{ab}$ is used as a shared key between Alice and Bob.
  - DDH assumption $\Rightarrow K$ is indistinguishable from a random key
  - K is a master key which is used to encrypt session keys. Session keys are used to encrypt traffic with a symmetric cryptosystem

Diffie-Hellamn: security

- A *(passive)* adversary
  - Knows Z_p^*, g
  - Sees g^a, g^b
  - Wants to compute g^{ab}, or at least learn something about it
- Recall the Decisional Diffie-Hellman problem:
  - Given random $x,y \in Z_p^*$, such that $x=g^a$ and $y=g^b$; and a value z which is promised to be either g^{ab} or g^c (for a random c), it is hard tell which is the case.
  - I.e., g^{ab} is indistinguishable from a random element in *H.*

  - *Note:* it is insufficient to require that the adversary cannot compute g^{ab}.

Diffie-Hellman key exchange: usage

- The DH key exchange can be used in any group *in which the Decisional Diffie-Hellman (DDH) assumption is believed to hold.*
- Currently, Z_p^* and elliptic curve groups.

- Common usage:
  - Overhead: 1-2 exponentiations
  - Usually,
    - A DH key exchange for generating a master key
    - Master key used to encrypt session keys
    - Session key is used to encrypt traffic with a symmetric cryptosystem

An active attack against the Diffie-Hellman Key Exchange Protocol

- An active adversary Eve.
- Can read and change the communication between Alice and Bob.
- …As if Alice and Bob communicate via Eve.

Alice ←→ Eve ←→ Bob

Man –in-the-Middle: an active attack against the Diffie-Hellman Key Exchange protocol

- Alice:
  - picks a random $a \in [1,q]$.
  - Sends g^a mod p to Bob.

- Bob:

Eve changes g^a to g^c →

  - picks a random $b \in [1,q]$.
  - Sends g^b mod p to Alice.

← Eve changes g^b to g^d

  - *Computes $k=(g^d)^a$ mod p*

  - *Computes $k=(g^c)^b$ mod p*

Keys:		
Alice	Eve	Bob
g^{ad}	g^{ad}, g^{bc}	g^{bc}

- Solution: ?

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