# Introduction to Cryptography
# Lecture 12

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- Some practical issues in number theory

- Last week
  - Primality testing
  - Pollard’s rho method for factoring

# Integer factorization

- The RSA and Rabin cryptosystems use a modulus $N$ and are insecure if it is possible to factor $N$.
- Factorization: given $N$ find all prime factors of $N$.
- Factoring is the search problem corresponding to the primality testing decision problem.
  - Primality testing is easy
  - What about factoring?

# Pollard's Rho method

- Factoring $N$
- Trivial algorithm: trial division by all integers < $N^{1/2}$.
- Pollard's rho method:
  - $O(N^{1/4})$ computation.
  - O(1) memory.
  - A heuristic algorithm.

# Modern factoring algorithms

- The number-theoretic running time function $L_n(a,c)$

$$L_n(a,c) = e^{c(\ln n)^a (\ln\ln n)^{1-a}}$$

  - For a=0, the running time is polynomial in ln(n).
  - For a=1, the running time is exponential in ln(n).
  - For 0<a<1, the running time is subexponential.

- Factoring algorithms
  - Quadratic field sieve: $L_n(1/2, 1)$
  - General number field sieve: $L_n(1/3, 1.9323)$
  - Elliptic curve method $L_p(1/2, 1.41)$ (preferable only if p<<sqrt(n) )

# Modulus size recommendations

- Factoring algorithms are run on massively distributed networks of computers (running in their idle time).
- RSA published a list of factoring challenges.
- A 512 bit challenge was factored in 1999.
- The largest factored number $n=pq$.
  - 768 bits (RSA-768)
  - Factored on January 7, 2010 using the NFS
- Typical current choices:
  - At least 1024-bit RSA moduli should be used
  - For better security, longer RSA moduli are used
  - For more sensitive applications, key lengths of 2048 bits (or higher) are used

# RSA with a modulus with more factors

- The best factoring algorithms:
  - General number field sieve (NFS): $L_n(1/3, 1.9323)$
  - Elliptic curve method $L_p(1/2, 1.41)$

- If n=pq, where |p|=|q|, then the NFS is faster.
  - This is true even though $p=n^{1/2}$.
  - Common parameters: |p|=|q|=512 bits
  - Factoring using the NFS is infeasible, but more likely than factoring using the elliptic curve method.

# RSA for paranoids

- Suppose $N=pq$, $|p|=500$ bits, $|q|=4500$ bits.
- Factoring is extremely hard.
  - The NFS has to be applied to a much larger modulus. The elliptic curve method is still inefficient.
- Decryption is also very slow. (Encryption is done using a short exponent, so it is pretty efficient.)

- However, in most applications RSA is used to transfer session keys, which are rather short.
- Assume message length is < 500 bits.
  - In the decryption process, it is only required to decrypt the message mod p. (More efficient than mod a 1024 bit n.)
  - Encryption must use a slightly longer *e*. Say, *e*=20.

# Discrete log algorithms

- Input: *(g,y)* in a finite group G. Output: x s.t. $g^x = y$ in G.
- Generic vs. special purpose algorithms: generic algorithms do not exploit the representation of group elements.

- Algorithms
  - Baby-step giant-step: Generic. |G| can be unknown. Sqrt(|G|) running time and memory.
  - Pollard's rho method: Generic. |G| must be known. Sqrt(|G|) running time and O(1) memory.
  - No generic algorithm can do better than O(sqrt(q)), where q is the largest prime factor of |G|
  - Pohlig-Hellman: Generic. |G| and its factorization must be known. O(sqrt(q) ln q), where q is largest prime factor of |G|.
  - Therefore for $Z^*_p$, p-1 must have a large prime factor.
  - Index calculus algorithm for $Z^*_p$: L(1/2, c)
  - Number field size for $Z^*_p$: L(1/3, 1.923)

# Elliptic Curves

- The best discrete log algorithm which works even if |G| can be unknown is the baby-step giant-step algorithm.
  - Sqrt(|G|) running time and memory.
- Other (more efficient) algorithms must know |G|.
  - In $Z_p^*$ we know that | $Z_p^*$ |=p-1.

- Elliptic curves are groups G where
  - The Diffie-Hellman assumption is assumed to hold, and therefore we can run DH an ElGamal encryption/sigs.
  - |G| is unknown and therefore the best discrete log algorithm us pretty slow
  - It is therefore believed that a small Elliptic Curve group is as secure as larger $Z_p^*$ group.
  - Smaller group -> smaller keys and more efficient operations.

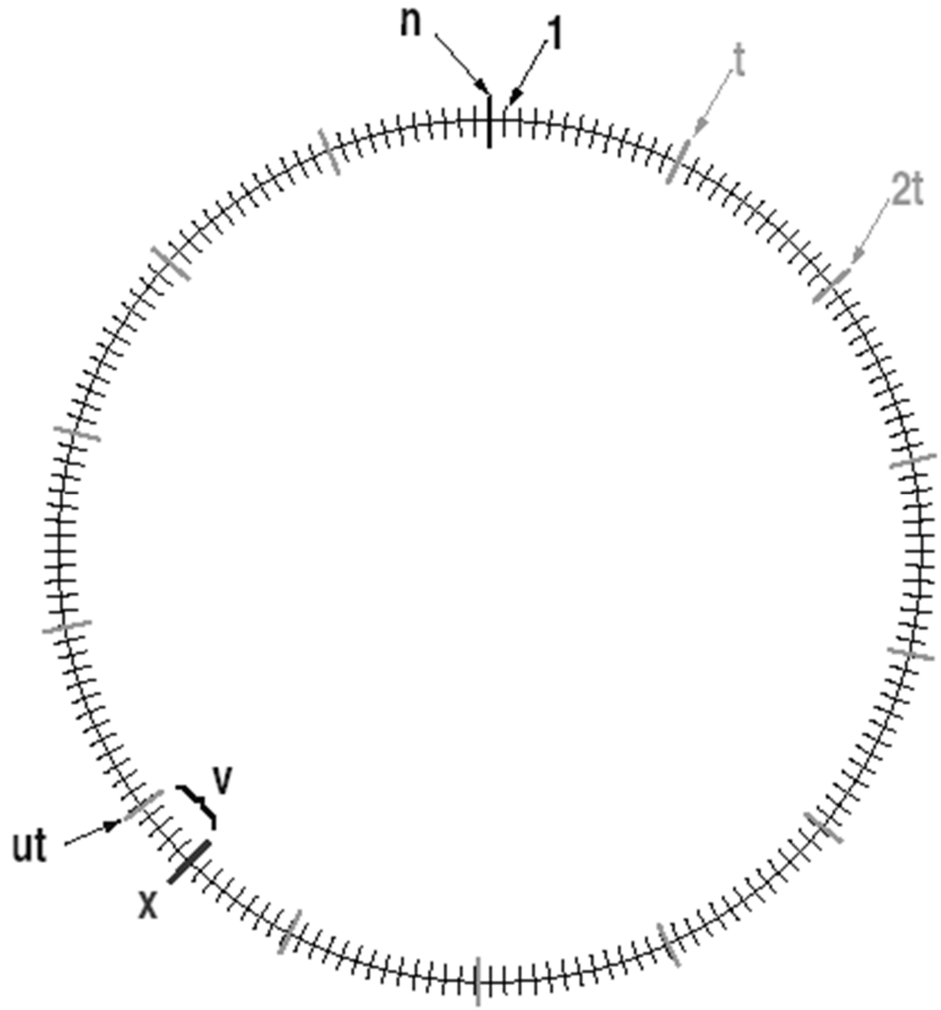
# Baby-step giant-step DL algorithm

- Let t=sqrt(|G|).
- x can be represented as $x=ut-v$, where u,v < sqrt(|G|).

- The algorithm:
  - Giant step: compute the pairs $(j, g^{j \cdot t})$, for $0 \le j \le t$. Store in a table keyed by $g^{j \cdot t}$.
  - Baby step: compute $y \cdot g^i$ for $i=0,1,2\ldots$, until you hit an item $(j, g^{j \cdot t})$ in the table. x $= jt - i$.

- Memory and running time are O(sqrt|G|).

# Baby-step giant-step DL algorithm



# Secret sharing

# Secret Sharing

- 3-out-of-3 secret sharing:
  - Three parties, A, B and C.
  - Secret *S.*
  - No two parties should know *anything* about *S*, but all three together should be able to retrieve it.
- In other words
  - $A + B + C \Rightarrow S$
  - But,
    - $A + B \not\Rightarrow S$
    - $A + C \not\Rightarrow S$
    - $B + C \not\Rightarrow S$

# Secret Sharing

- 3-out-of-3 secret sharing:
- How about the following scheme:
  - Let $S=s_1s_2\ldots s_m$ be the bit representation of $S$. ($m$ is a multiple of 3)
    - Party A receives $s_1,\ldots,s_{m/3}$.
    - Party B receives $s_{m/3+1},\ldots,s_{2m/3}$.
    - Party C receives $s_{2m/3+1},\ldots,s_m$.
  - All three parties can recover $S$.

  - Why doesn't this scheme satisfy the definition of secret sharing?
  - Why does each share need to be as long as the secret?

# Secret Sharing

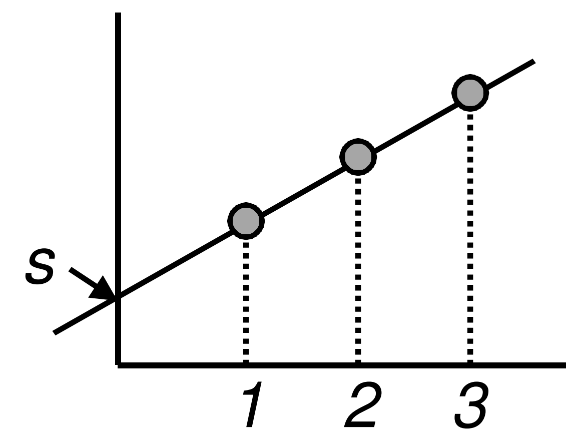
- Solution:
  - Define *shares* for A,B,C in the following way
  - $(S_A, S_B, S_C)$ is a random triple, subject to the constraint that
    - $S_A \oplus S_B \oplus S_C = S$
    - *or,* $S_A$ and $S_B$ are random, and $S_C = S_A \oplus S_B \oplus S$.

- What if it is required that any one of the parties should be able to compute $S$?
  - Set $S_A = S_B = S_C = S$

- What if each pair of the three parties should be able to compute $S$?

# *t*-out-of-*n* secret sharing

- Provide shares to *n* parties, satisfying
  - Recoverability: any *t* shares enable the reconstruction of the secret.
  - Secrecy: any *t-1* shares reveal nothing about the secret.
- We saw *1*-out-of-*n* and *n*-out-of-*n* secret sharing.
- Consider *2*-out-of-*n* secret sharing.
  - Define a line which intersects the Y axis at *S*
  - The shares are points on the line
  - Any two shares define *S*
  - A single share reveals nothing



# *t*-out-of-*n* secret sharing

- Fact: Let $F$ be a field. Any $d+1$ pairs $(a_i , b_i )$ define a unique polynomial $P$ of degree ≤ $d$, s.t. $P(a_i )=b_i$. (assuming $d < |F|$).

- Shamir's secret sharing scheme:
  - Choose a large prime and work in the field $Zp$.
  - The secret S is an element in the field.
  - Define a polynomial $P$ of degree $t-1$ by choosing random coefficients $a_1,\ldots,a_{t-1}$ and defining

    $P(x) = a_{t-1}x^{t-1}+\ldots+a_1x+\underline{S}$.
  - The share of party $j$ is $( j, P(j) )$.

# $t$-out-of-$n$ secret sharing

- Reconstruction of the secret:
  - Assume we have $P(x_1),\ldots,P(x_t)$.
  - Use Lagrange interpolation to compute the unique polynomial of degree ≤ $t-1$ which agrees with these points.
  - Output the free coefficient of this polynomial.

- Lagrange interpolation
  - $P(x) = \sum_{i=1..t} P(x_i)\cdot L_i(x)$
  - where $L_i(x)=\prod_{j\neq i}(x-x_j) \,/\, \prod_{j\neq i}(x_i-x_j)$
  - (Note that $L_i(x_i)=1$, $L_i(x_j)=0$ for $j\neq i$.)

  - I.e., $S = \sum_{i=1..t} P(x_i) \cdot \prod_{j\neq i} -x_j \,/\, \prod_{j\neq i}(x_i - x_j)$

# Properties of Shamir's secret sharing

- Perfect secrecy: Any $t$-$1$ shares give no information about the secret: $\Pr(secret=s \mid P(1),\ldots,P(t-1)) = \Pr(secret=s)$. (Security is not based on any assumptions.)
- Proof:
  - Let's get intuition from 2-out-of-n secret sharing
  - The polynomial is generated by choosing a random coefficient $a$ and defining $P(x)= a\cdot x+s$.
  - Suppose that the adversary knows $P(x_1)=a\cdot x_1+s$.
  - For any value of $s$, there is a one-to-one and onto correspondence between $a$ and $P(x_1)$.
  - Since $a$ is uniformly distributed, so is the value of $P(x_1)$ (any assignment to $a$ results in exactly one value of $P(x_1)$).
    - Therefore $P(x_1)$ does not reveal any information about $s$.

# Properties of Shamir's secret sharing

- Perfect secrecy: Any *t-1* shares give no information about the secret: $\Pr(secret=s \mid P(1),\ldots,P(t-1)) = \Pr(secret=s)$. (Security is not based on any assumptions.)
- Proof:
  - The polynomial is generated by choosing a random polynomial of degree *t-1,* subject to $P(0)$=secret.
  - Suppose that the adversary knows the shares $P(x_1),\ldots,P(x_{t-1})$.
  - The values of $P(x_1),\ldots,P(x_{t-1})$ are defined by *t-1* linear equations of $a_1,\ldots,a_{t-1}$, *s.*
    - $P(x_i) = \Sigma_{i=1,\ldots,t-1} ( x_i )^j a_j + s.$

# Properties of Shamir's secret sharing

- Proof (cont.):
  - The values of $P(x_1),\ldots,P(x_{t-1})$ are defined by $t-1$ linear equations of $a_1,\ldots,a_{t-1}, s$.
    - $P(x_i) = \Sigma_{j=1,\ldots,t-1} ( x_i )^j a_j + s.$
  - For any possible value of $s$, there is a exactly one set of values of $a_1,\ldots,a_{t-1}$ which gives the values $P(x_1),\ldots,P(x_{t-1})$.
    - This set of $a_1,\ldots,a_{t-1}$ can be found by solving a linear system of equations.
  - Since $a_1,\ldots,a_{t-1}$ *are* uniformly distributed, so are the values of $P(x_1),\ldots,P(x_{t-1})$.
    - Therefore $P(x_1),\ldots,P(x_{t-1})$ reveal nothing about $s$.

# Additional properties of Shamir's secret sharing

- Ideal size: Each share is the same size as the secret.
- Extendable: Additional shares can be easily added.
- Flexible: different weights can be given to different parties by giving them more shares.
- Homomorphic property: Suppose *P(1),…,P(n)* are shares of *S,* and *P'(1),…,P'(n)* are shares of *S'*, then *P(1)+P'(1),…,P(n)+P'(n)* are shares for *S+S'.*

# General secret sharing

- $P$ is the set of users (say, $n$ users).
- $A \in \{1,2,\ldots,n\}$ is an authorized subset if it is authorized to access the secret.
- $\Gamma$ is the set of authorized subsets.
- For example,
  - $P = \{1,2,3,4\}$
  - $\Gamma$ = *Any set containing one of* $\{ \{1,2,4\}, \{1,3,4,\}, \{2,3\} \}$
  - Not supported by threshold secret sharing

- If $A \in \Gamma$ and $A \subseteq B$, then $B \in \Gamma$.
- $A \in \Gamma$ is a minimal authorized set if there is no $C \subseteq A$ such that $C \in \Gamma$.
- The set of minimal subsets $\Gamma_0$ is called the basis of $\Gamma$.

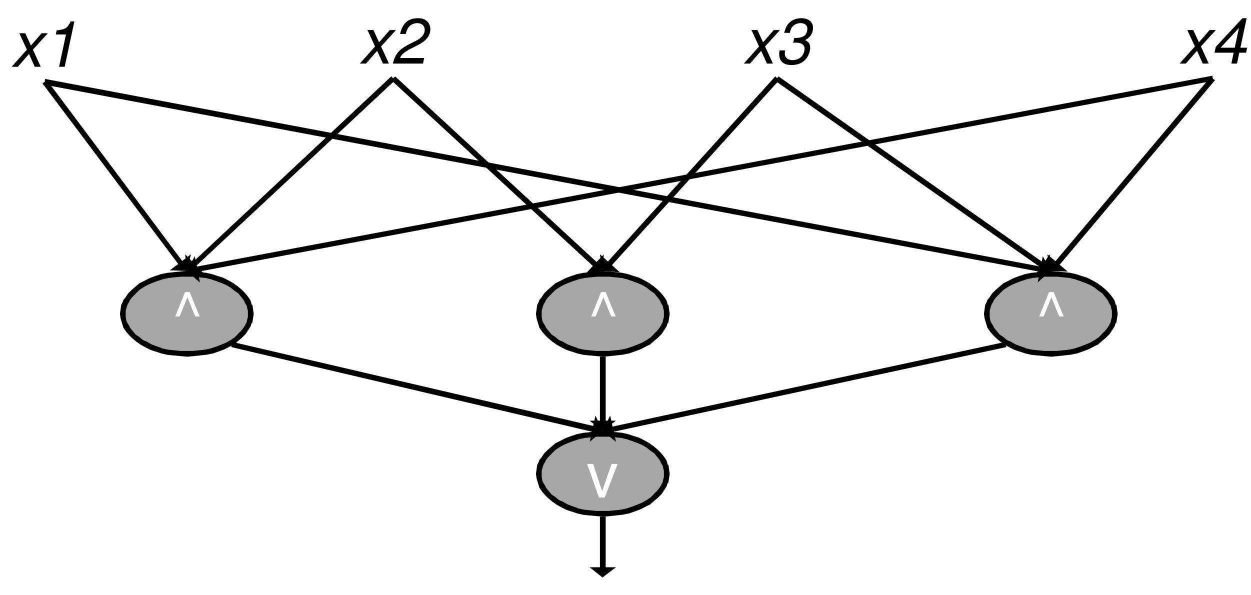
# Why should we examine general access structures?

- Some general access structures can be implemented using threshold access structures.
- But not all access structures can be represented by threshold access structures

- For example, consider the access structure $\Gamma=\{\{1,2\},\{3,4\}\}$
  - Any threshold based secret sharing scheme with threshold t gives weights to parties, such that $w_1+w_2 \geq t$, and $w_3+w_4 \geq t$.
  - Therefore either $w_1 \geq t/2$, or $w_2 \geq t/2$. Suppose that this is $w_1$.
  - Similarly either $w_3 \geq t/2$, or $w_4 \geq t/2$. Suppose that this is $w_3$.
  - In this case parties 1 and 3 can reveal the secret, since $w_1+w_3 \geq t$.
  - Therefore, this access structure cannot be realized by a threshold scheme.

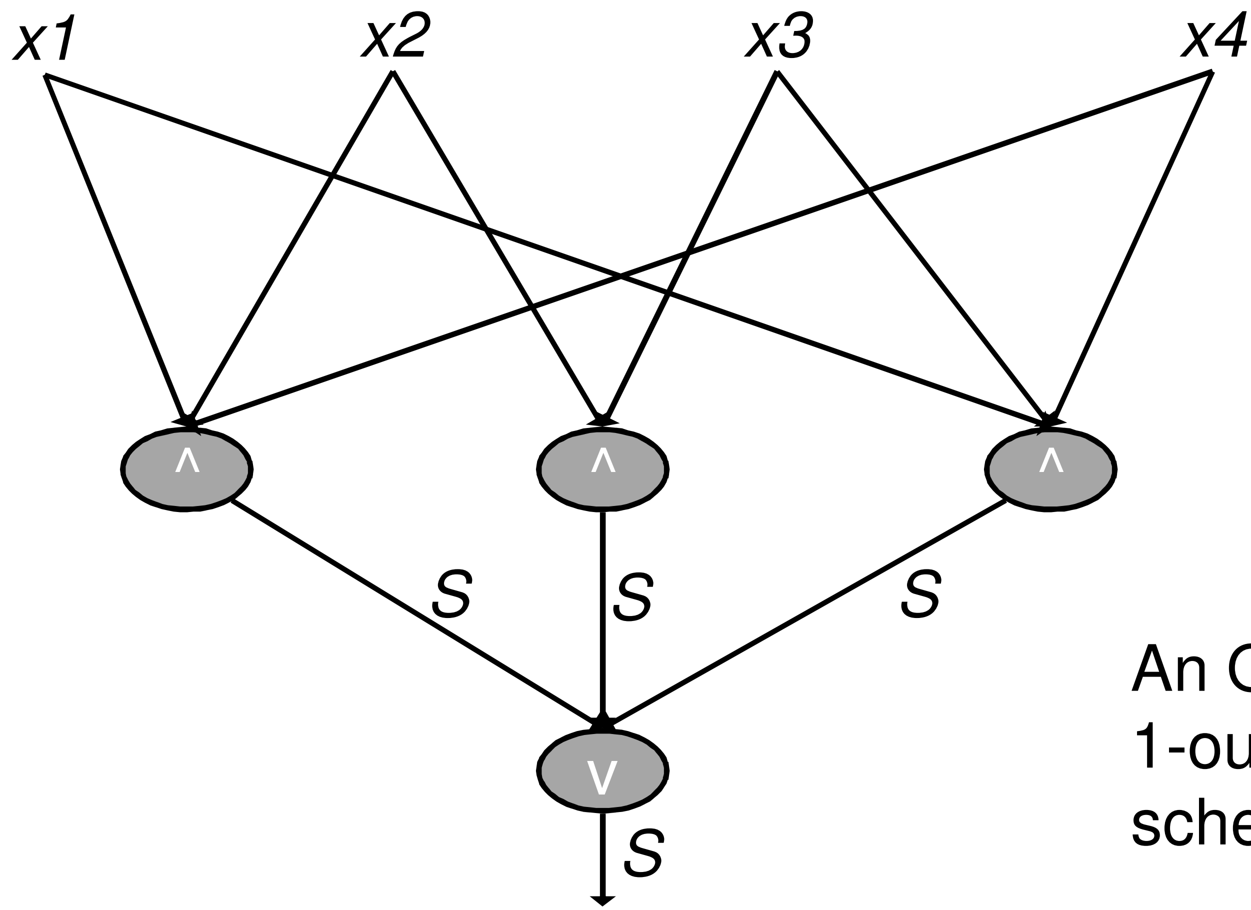
# The monotone circuit construction (Benaloh-Leichter)

- Given $\Gamma$ construct a circuit C s.t. *C(A)=1* iff A$\in\Gamma$.
  - $\Gamma_0 = \{ \{1,2,4\}, \{1,3,4,\}, \{2,3\} \}$
- This Boolean circuit can be constructed from OR and AND gates, and is *monotone.* Namely, if *C(x)=1*, then changing bits of *x* from 0 to 1 doesn't change the result to 0.



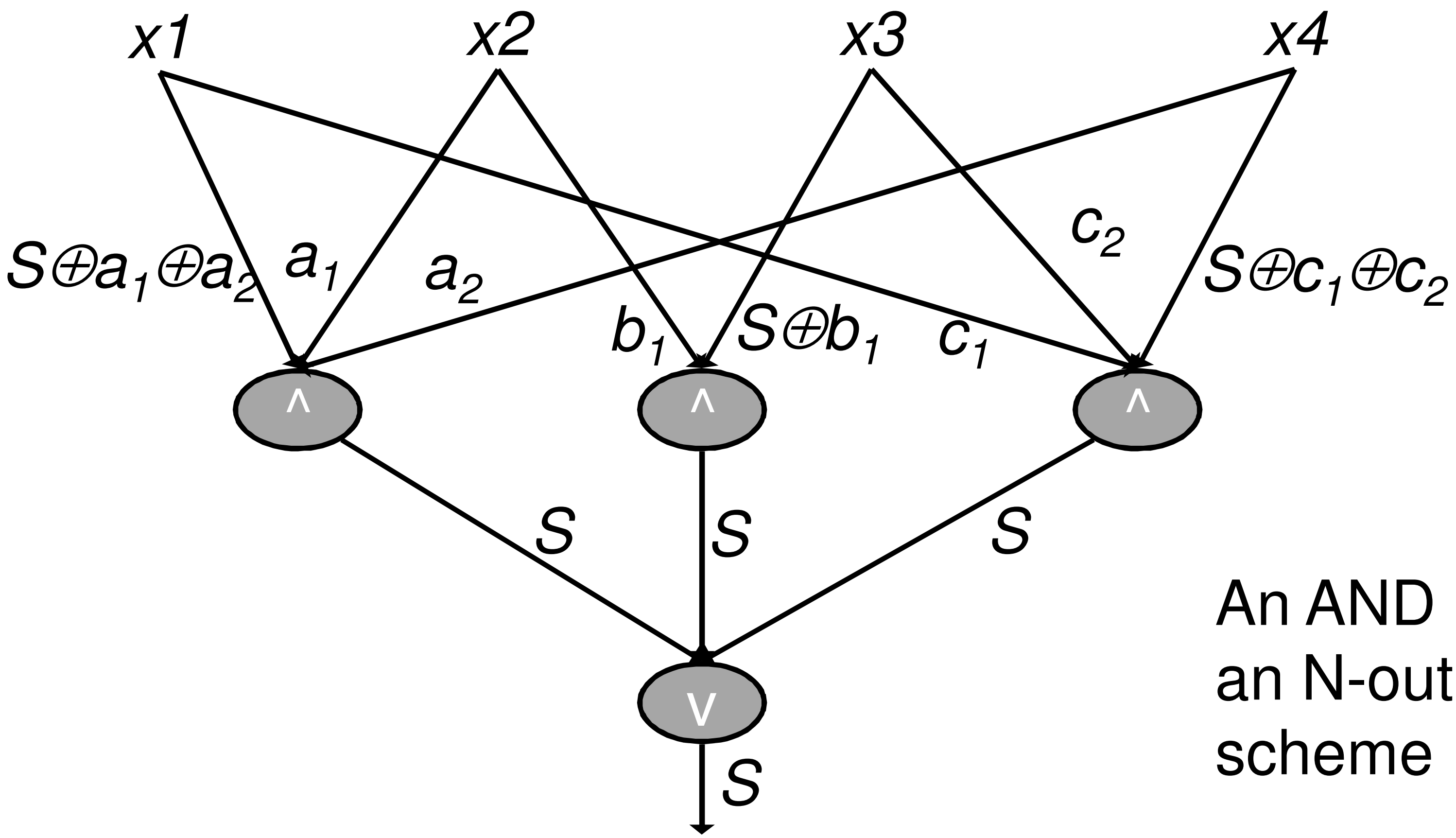
# Handling OR gates

Starting from the output gate and going backwards




An OR gate is a 1-out-of-N scheme

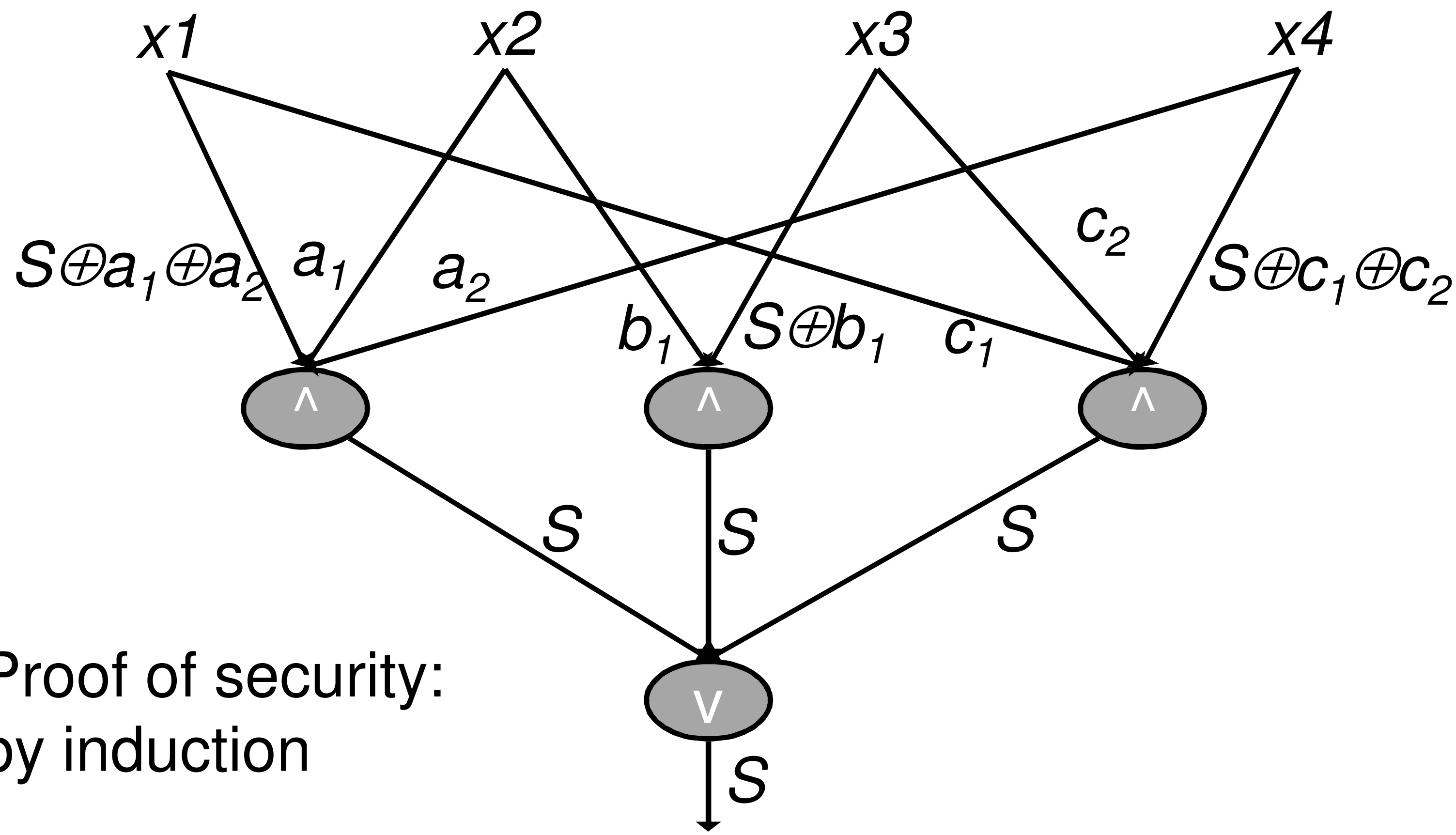
# Handling AND gates




An AND gate is an N-out-of-N scheme

# Handling AND gates

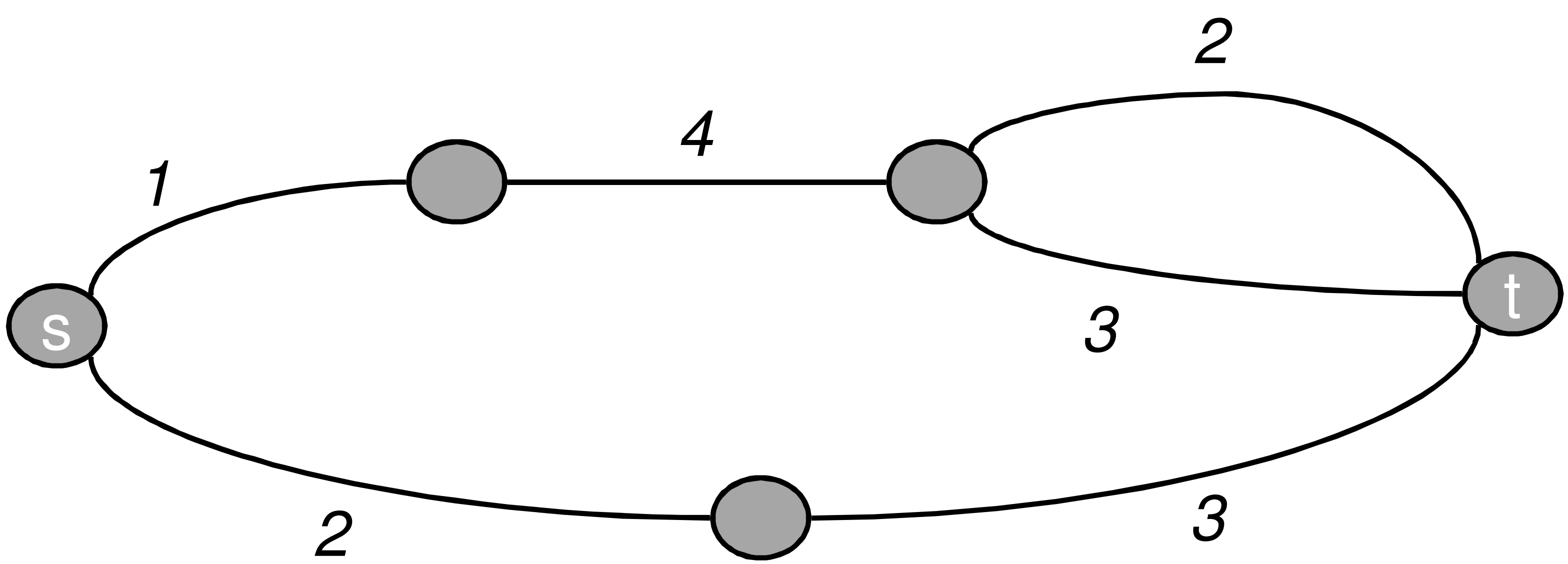
Final step: each user gets the keys of the wires going out from its variable




Proof of security: by induction

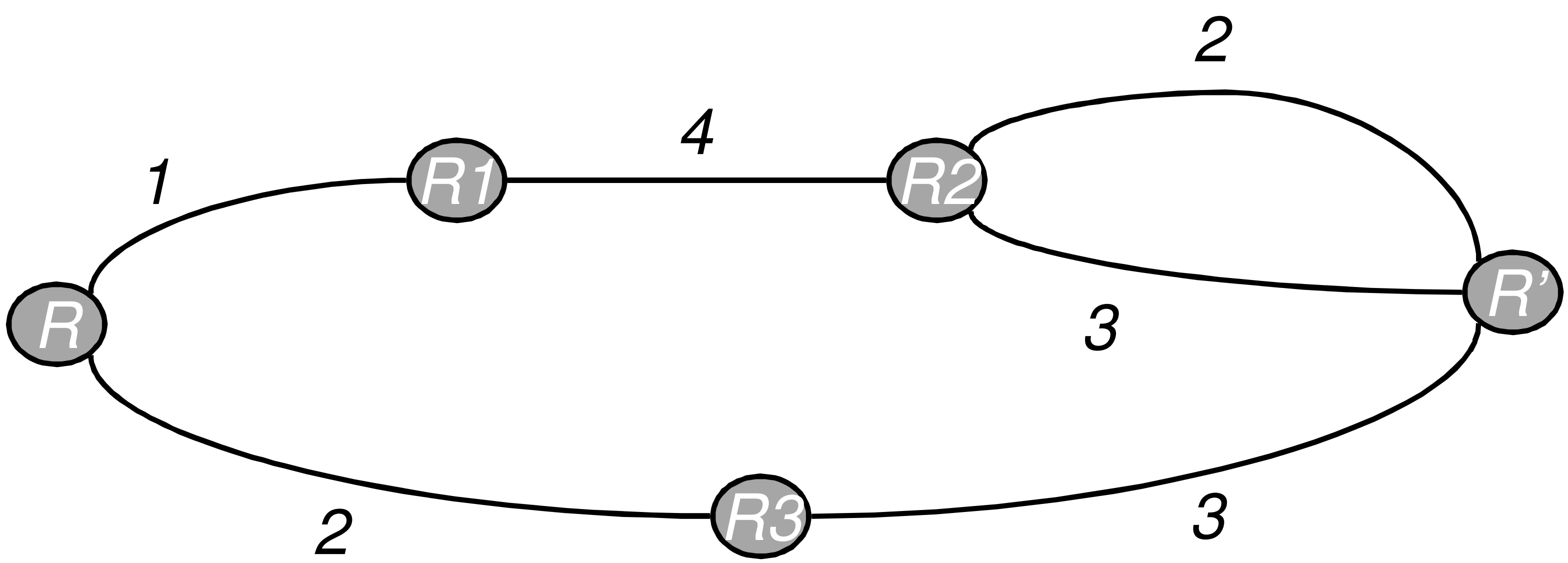
# A graph based construction

- Represent the access structure by an undirected graph.
- An authorized set corresponds to a path from s to t in an undirected graph.
- $\Gamma_0 = \{\ \{1,2,4\}, \{1,3,4,\}, \{2,3\}\ \}$

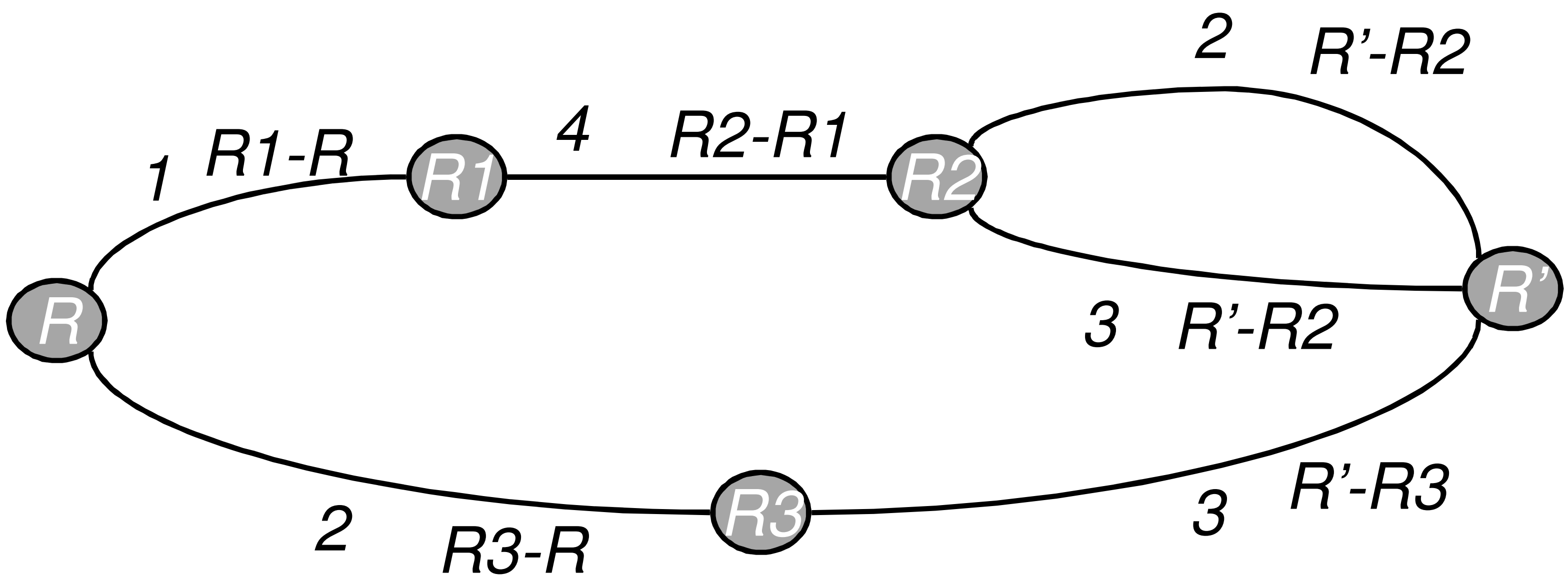


# A graph based construction

Assign random values to nodes, s.t. *R'-R= shared secret* *(R'=R+shared secret)*



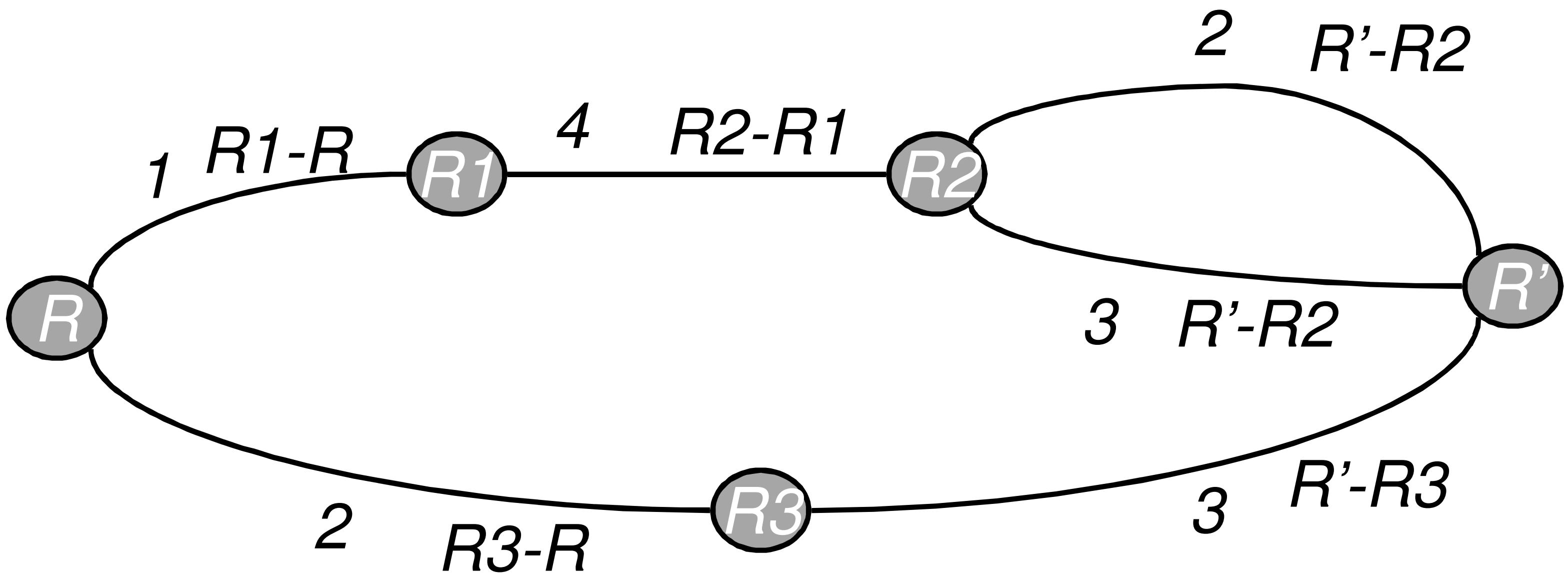
# A graph based construction




- Assign to edge R1→R2 the value *R2-R1*
- Give to each user the values associated with its edges

# A graph based construction

- Consider the set {1,2,4}
- why can an authorized set reconstruct the secret? Why can't a unauthorized set do that?



# Electronic cash

# Simple electronic checks

- A payment protocol:
  – Sign a document transferring money from your account to another account
  - This document goes to your bank
  - The bank verifies that this is not a copy of  a previous check
  - The bank checks your balance
  - The bank transfers the sum

- Problems:
  - Requires online access to the bank (to prevent reusage)
  - Expensive.
  - The transaction is traceable (namely, the bank knows about the transaction between you and Alice).

# First try at a payment protocol

- Withdrawal
  - User gets bank signature on {I am a $100 bill, #1234}
  - Bank deducts $100 from user's account
- Payment
  - User gives the signature to a merchant
  - Merchant verifies the signature, and checks online with the bank to verify that this is the first time that it is used.
- Problems:
  - As before, online access to the bank, and lack of anonymity.
- Advantage:
  - The bank doesn't have to check online whether there is money in the user's account.
  - In fact, there is no real need for the signature, since the bank checks its own signature.

# Anonymous cash via blind signatures

- In order to preserve payer's anonymity the bank signs the bill without seeing it
  - (e.g. like signing on a carbon paper)

- RSA Blind signatures (Chaum)
- RSA signature: $(H(m))^{1/e}$ mod $n$
- Blind RSA signature:
  - Alice sends Bob ($r^e\, H(m)$) mod $n$, where $r$ is a random value.
  - Bob computes $(r^e\, H(m))^{1/e} = r\, H(m)^{1/e}$ mod $n$, and sends to Alice.
  - Alice divides by $r$ and computes $H(m)^{1/e}$ mod $n$

- Problem: Alice can get Bob to sign anything, Bob does not know what he is signing.

# Enabling the bank to verify the signed value

- "cut and choose" protocol
- Suppose Alice wants to sign a $20 bill.
  - A $20 bill is defined as *H(random index,$20)*.
  - Alice sends to bank 100 different $20 bills for blind signature.
  - The bank chooses 99 of these and asks Alice to unblind them (divide by the corresponding *r* values). It verifies that they are all $20 bills.
  - The bank blindly signs the remaining bill and gives it to Alice.
  - Alice can use the bill without being identified by the bank.

- If Alice tries to cheat she is caught with probability 99/100.
- 100 can be replaced by any parameter *m*.
- But we would like to have an exponentially small cheating probability.

# Exponentially small cheating probability

- Define that a \$20 bill in a new way:
  - The bill is valid if it is the RSA signature of the multiplication of 50 values of the form *H(x),* (where *x="random index,\$20"*).

- The withdrawal protocol:
  - Alice sends to the Bank $z_1, z_2, \ldots, z_{100}$ (where $z_i = r_i^{\,e} \cdot H(x_i)$).
  - The Bank asks Alice to reveal ½ of the values $z_i = r_i^{\,e} \cdot H(x_i)$.
  - The Bank verifies them and extracts the $e^{th}$ root of the multiplication of all the other 50 values. Alice divides the results by the multiplication of the corresponding $r_i$ values.
- Payment: Alice sends the signed bill and reveals the 50 preimage values. The merchant sends them to the bank which verifies that they haven't been used before.

- Alice can only cheat if she guesses the 50 locations in which she will be asked to unblind the $z_i$s, which happens with probability $\sim 2^{-100}$.

# Online vs. offline digital cash

- We solved the anonymity problem, while verifying that Alice can only get signatures on bills of the right value.

- The bills can still be duplicated
- Merchants must check with the bank whenever they get a new bill, to verify that it wasn't used before.

- A new idea:
  - During the payment protocol the user is forced to encode a random identity string (RIS) into the bill
  - If the bill is used twice, the RIS reveals the user's identity and she loses her anonymity.

# Offline digital cash

Withdrawal protocol:

- Alice prepares 100 bills of the form
  - {I am a $20 bill, *#1234*, $y_1, y'_1, y_2, y'_2, \ldots, y_m, y'_m$}
  - S.t. $\forall i\ y_i = H(x_i)$, $y'_i = H(x'_i)$, and it holds that $x_i \oplus x'_i$ = *Alice's id,* where *H()* is a collision resistant function.
- Alice blinds these bills and sends to the bank.
- The bank asks her to unblind 99 bills and show their $x_i, x'_i$ values, and checks their validity.
  - (Alternatively, as in the previous example, Alice can do a check with fails with only an exponential probability.)
- The bank signs the remaining blinded bill.

# Offline digital cash

Payment protocol:

- Alice gives a signed bill to the vendor
  - {I am a \$20 bill, *#1234*, $y_1, y'_1, y_2, y'_2, \ldots, y_m, y'_m$}
- The vendor verifies the signature, and if it is valid sends to Alice a random bit string b=$b_1 b_2 \ldots b_m$ of length *m.*
- $\forall$ *i* if $b_i=0$ *Alice returns* $x_i$, otherwise ($b_i=1$) she returns $x'_I$
- The vendor checks that $y_i=H(x_i)$ or $y'_i=H(x'_i)$ (depending on $b_i$ *).* If this check is successful it accepts the bill. (Note that Alice's identity is kept secret.)

- Note that the merchant does not need to contact the bank during the payment protocol.

# Offline digital cash

- The merchant must deposit the bill in the bank. It cannot use the bill to pay someone else.
  - Because it cannot answer challenges $b^*$ different than the challenge $b$ it sent to Alice.

- How can the bank detect double spenders?
  - Suppose two merchants $M$ and $M^*$ receive the same bill
  - With very high probability, they ask Alice *different* queries $b,b^*$
  - There is an index $i$ for which $b_i=0$, $b^*_i=1$. Therefore $M$ receives $x_i$ and $M^*$ receives $x'_i$.
  - When they deposit the bills, the bank receives $x_i$ and $x^*_i$, and can compute $x_i \oplus x'_i$ =*Alice's id.*