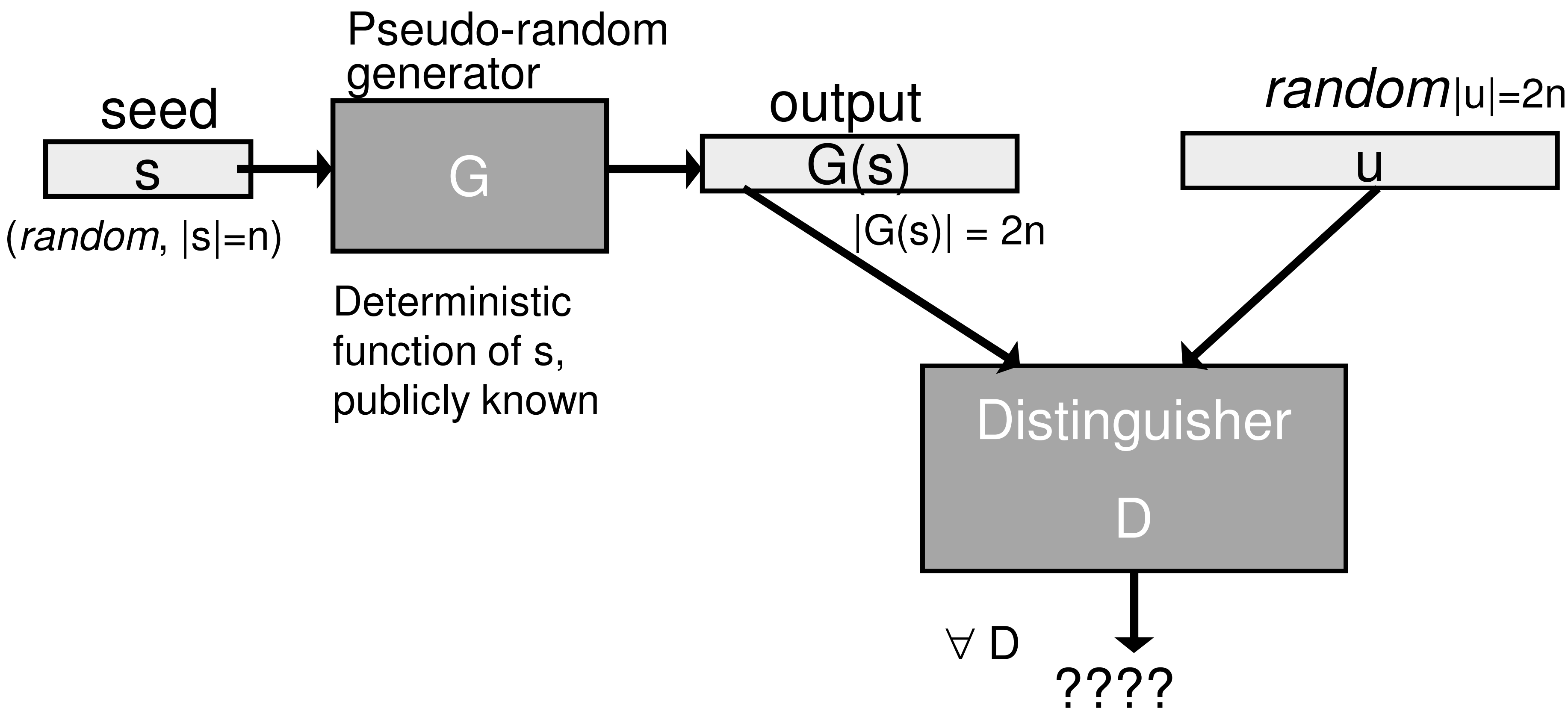
Introduction to Cryptography

Lecture 3

Benny Pinkas

Pseudo-random generator



Pseudo-random generators

- Pseudo-random generator (PRG)
  - G: $\{0,1\}^n \Rightarrow \{0,1\}^m$
    - A deterministic function, computable in polynomial time.
    - It must hold that m > n. Let us assume m=2n.
    - The function has only 2^n possible outputs.
- Pseudo-random property:
  - $\forall$ polynomial time adversary D, (whose output is 0/1)

    if we choose inputs $s \in_R \{0,1\}^n$, $u \in_R \{0,1\}^m$, (in other words, choose s and u uniformly at random), then

    it holds that D(G(s)) is similar to D(u)

    namely, | Pr[D(G(s))=1] - | Pr[D(u)=1] | is negligible

Do PRGs exist?

- If P=NP then PRGs do not exist (why?)
- So their existence can only be conjectured until the P=NP question is resolved.

Using a PRG for Encryption

- Replace the one-time-pad with the output of the PRG

- Key: a (short) random key $k \in \{0,1\}^{|k|}$.
- Message $m = m_1,\ldots,m_{|m|}$.
- Use a PRG $G : \{0,1\}^{|k|} \rightarrow \{0,1\}^{|m|}$
- Key generation: choose $k \in \{0,1\}^{|k|}$ uniformly at random.
- Encryption:
  - Use the output of the PRG as a one-time pad. Namely,
  - Generate $G(k) = g_1,\ldots,g_{|m|}$
  - Ciphertext $C = g_1 \oplus m_1,\ldots, g_{|m|} \oplus m_{|m|}$

- This is an example of a *stream cipher.*

Security of encryption against polynomial adversaries

- Perfect security (previous equivalent defs):
  - (indistinguishability) $\forall\ m_0, m_1 \in M$, $\forall c$, the probability that c is an encryption of m_0 is equal to the probability that c is an encryption of m_1.
  - (semantic security) The distribution of m given the encryption of m is the same as the a-priori distribution of m.
- Security of pseudo-random encryption (equivalent defs):
  - (indistinguishability) $\forall\ m_0, m_1 \in M$, no *polynomial time* adversary D can distinguish between the encryptions of m_0 and of m_1. Namely, $\Pr[D(E(m_0))=1] \approx \Pr[D(E(m_1))=1)$
  - (semantic security) $\forall\ m_0, m_1 \in M$, a polynomial time adversary which is given $E(m_b)$, where $b \in_r \{0,1\}$, succeeds in finding b with probability $\approx$ ½.

Proofs by reduction

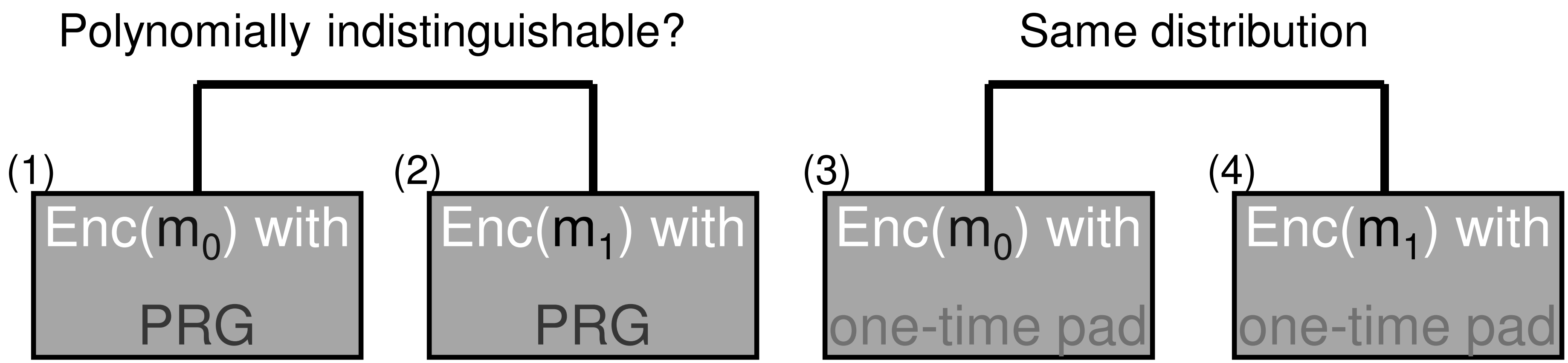
- We don't know how to prove unconditional proofs of computational security; we must rely on assumptions.
  - We can simply assume that the encryption scheme is secure. This is bad.
  - Instead, we will assume that some low-level problem is hard to solve, and then prove that the cryptosystem is secure under this assumption.
  - (For example, the assumption might be that a certain function G is a pseudo-random generator.)
  - Advantages of this approach:
    - It is easier to design a low-level function.
    - There are (very few) "established" assumptions in cryptography, and people prove the security of cryptosystem based on these assumptions.

Using a PRG for Encryption: Security

- The output of a pseudo-random generator is used instead of a one-time pad.
- Proof of security by reduction:
  - The assumption is that the PRG is strong (its output is indistinguishable from random).
  - We want to prove that in this case the encryption is strong (it satisfies the indistinguishability definition above).

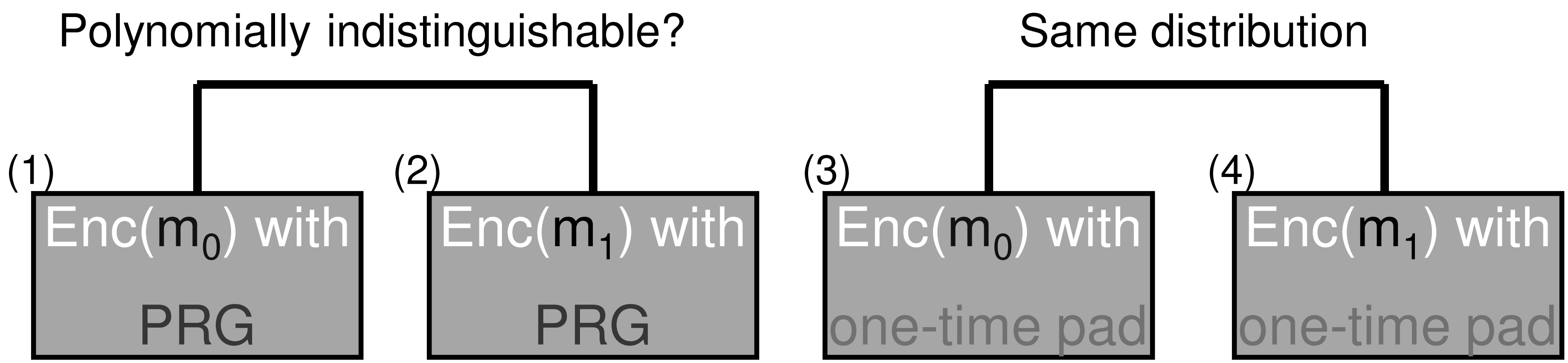
  - In other words, prove that if one can break the security of the encryption (distinguish between encryptions of m_0 and of m_1), then it is also possible to break the security of the PRG (distinguish its output from random).

Proof of Security



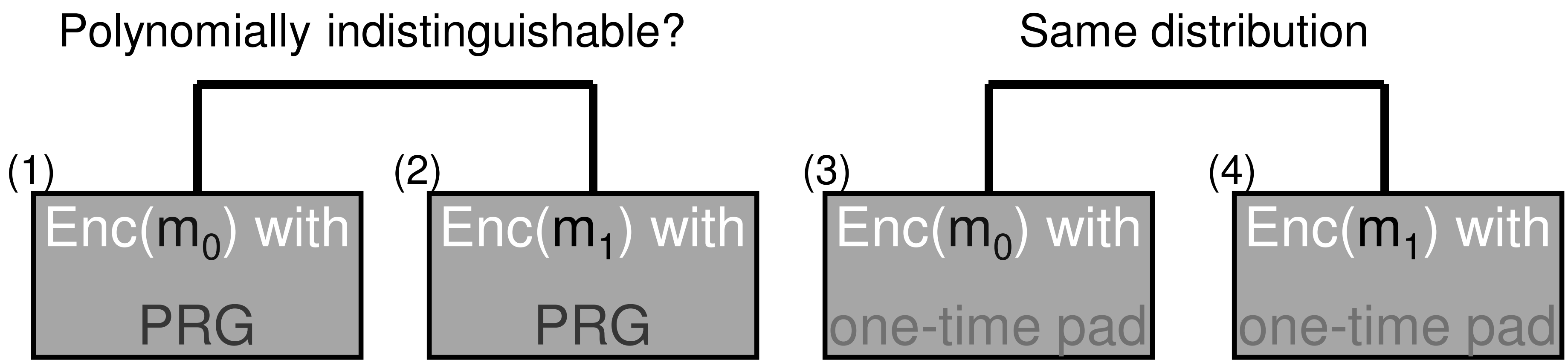
- Suppose that there is a distinguisher algorithm D'() which distinguishes between (1) and (2)
- We know that no D'() can distinguish between (3) and (4)
- We are given a string S and need to decide whether it is drawn from a pseudorandom distribution or from a uniformly random distribution
- We will use S as a pad to encrypt a message.

Proof of Security



- Recall: we assume that there is a D'() which always distinguishes between (1) and (2), and which distinguishes between (3) and (4) with probability ½.
- Choose a random $b \in \{0,1\}$ and compute $m_b \oplus S$. Give the result to D'().
  - if S was chosen uniformly, D'() must distinguish (3) from (4). (prob=½)
  - if S is pseudorandom, D'() must distinguish (1) from (2). (prob=1)
- If D'() outputs b then declare "pseudorandom", otherwise declare "random".
  - if S was chosen uniformly we output "pseudorandom" with prob ½.
  - if S is pseudorandom we output "pseudorandom" with prob 1.

Proof of Security



- Recall: we assume that there is a D'() which always distinguishes between (1) and (2), and which distinguishes between (3) and (4) with probability ½.
- Choose a random $b \in \{0,1\}$ and compute $m_b \oplus S$. Give the result to D'().
    - if S was chosen uniformly, D'() must distinguish (3) from (4). (prob=½)
    - if S is pseudorandom, D'() must distinguish (1) from (2). (prob=½+δ)
- If D'() outputs b then declare "pseudorandom", otherwise declare "random".
    - if S was chosen uniformly we output "pseudorandom" with prob ½.
    - if S is pseudorandom we output "pseudorandom" with prob ½+δ.

Stream ciphers

- Stream ciphers are based on pseudo-random generators.
  - Usually used for encryption in the same way as OTP
- Examples: A5, SEAL, RC4.
  - Very fast implementations.
  - RC4 is popular and secure when used correctly, but it was shown that its first output bytes are biased. This resulted in breaking WEP encryption in 802.11.

- Some technical issues:
  - Stream ciphers require *synchronization* (for example, if some packets are lost in transit).

RC4

- A stream cipher designed by Ron Rivest. Intellectual property belongs to RSA Inc.
  - Designed in 1987.
  - Kept secret until the design was leaked in 1994.

- Used in many protocols (SSL, WEP, etc.)

- Byte oriented operations.
- 8-16 machine operations per output byte.
- First output bytes are biased ☹

RC4 initialization

Word size is a single byte.

Input: $k_0;\dots;k_{255}$ (if key has fewer bits, pad it to itself sufficiently many times)

1. $j = 0$
2. $S_0 = 0;\ S_1 = 1;\dots;\ S_{255} = 255$
3. Let the key be $k_0;\dots;k_{255}$
4. For $i = 0$ to 255
   - $j = (j + S_i + k_i) \bmod 256$
   - Swap S_i and S_j

(note that S is a permutation of 0,…,255)

RC4 keying stream generation

An output byte `B` is generated as follows:

- $i = i + 1 \bmod 256$
- $j = j + S_i \bmod 256$
- Swap S_i and S_j
- $r = S_i + S_j \bmod 256$
- Output: $B = S_r$

`B` is xored to the next byte of the plaintext.

(since S is a permutation, we want that B is uniformly distributed)

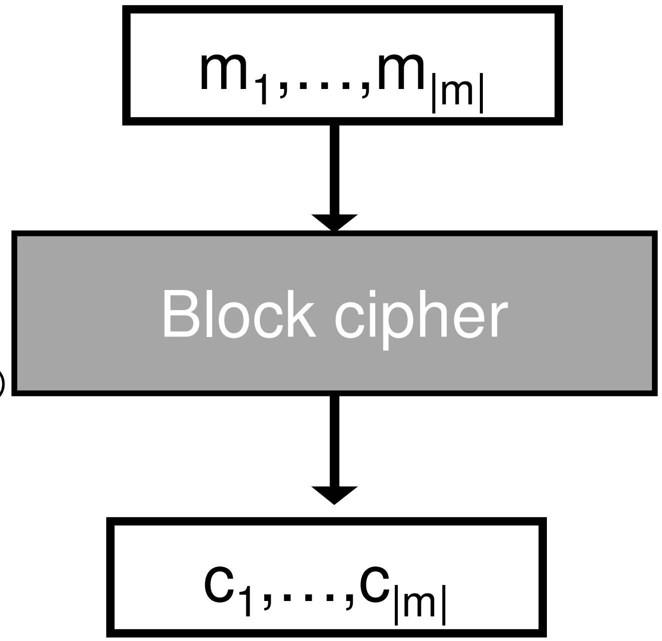
Bias: The probability that the first two output bytes are 0 is $2^{-16}+2^{-23}$ ☹

Block Ciphers

- Plaintexts, ciphertexts of fixed length, |m|. Usually, |m|=64 or 128 bits.
- The encryption algorithm E_k is a *permutation* over $\{0,1\}^{|m|}$, and the decryption D_k is its inverse. (They *are not* permutations of the bit order, but rather of the entire string.)

- Ideally, use a *random* permutation.
  – Implemented using a table with $2^{|m|}$ entries ☹
- Instead, use a *pseudo-random* permutation*, keyed by a key k.
  – Implemented by a computer program whose input is m,k.

  – (*) will be explained shortly

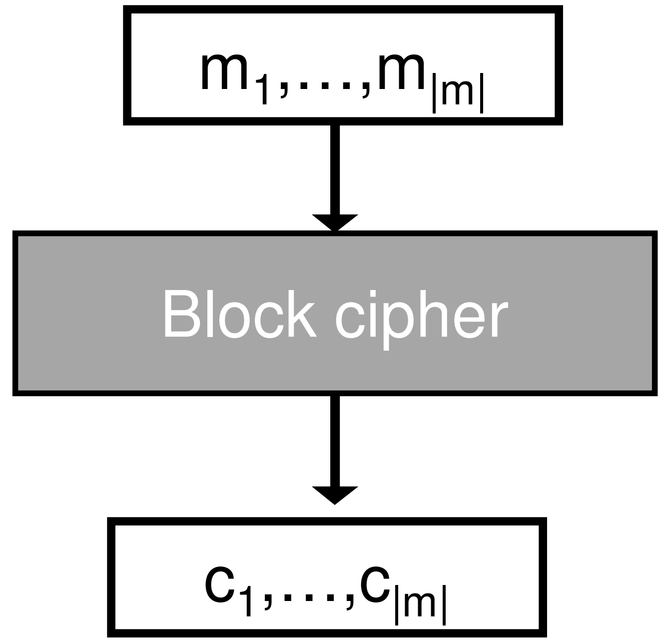


Block Ciphers

- Modeled as a pseudo-random permutation.

- Encrypt/decrypt whole blocks of bits
  - Might provide better encryption by simultaneously working on a block of bits
  - One error in ciphertext affects whole block
  - Delay in encryption/decryption
  - There was more research on the security of block ciphers than on the security of stream ciphers.
  - Avoids the synchronization problem of stream cipher usage.

- Different *modes of operation* (for encrypting longer inputs)



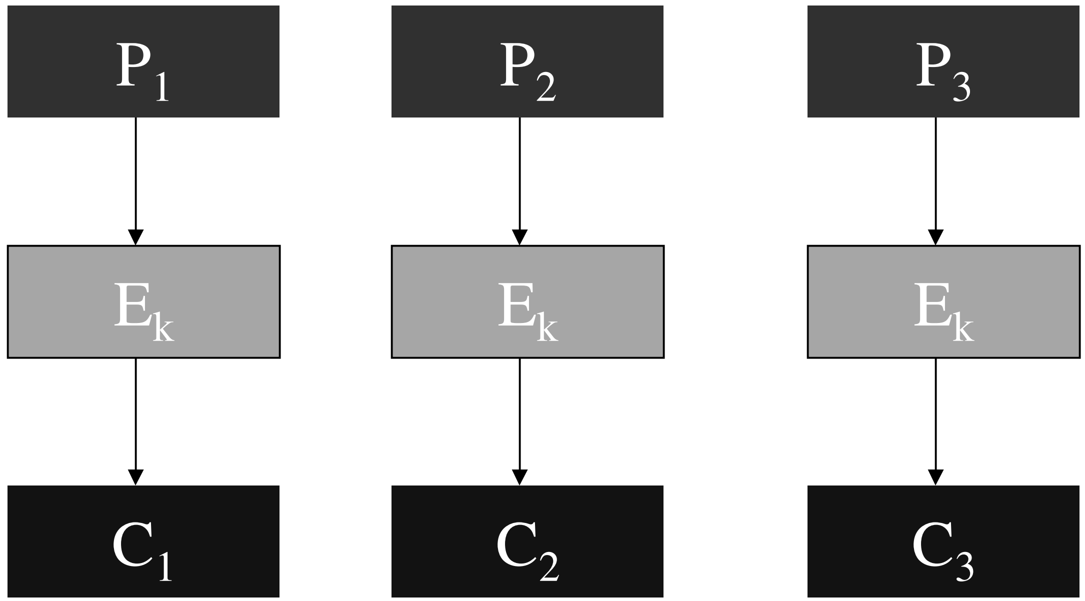
Block ciphers

- A block cipher is a function $F_k(x)$ of a key k and an |m| bit input x. It has an |m| bit output.
  - $F_k(x)$ is a keyed permutation

- How can we encrypt plaintexts longer than |m|?

- Different modes of operation were designed for this task.

ECB Encryption Mode (Electronic Code Book)



Namely, encrypt each plaintext block separately.

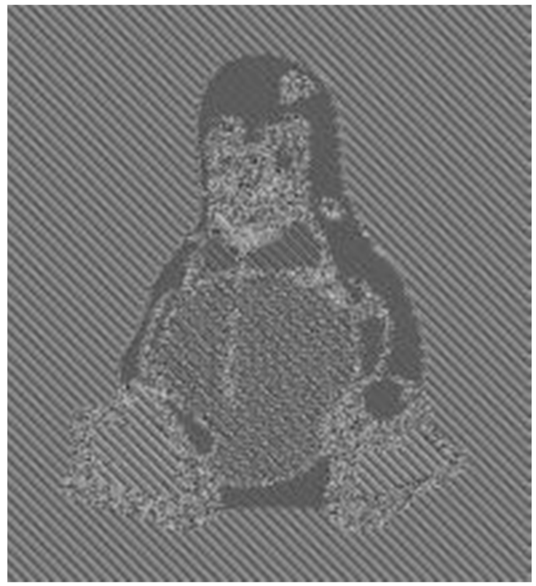
Properties of ECB

- Simple and efficient ☺
- Parallel implementation is possible ☺
- Does not conceal plaintext patterns ☹
  - Enc(P_1, P_2, P_1, P_3)

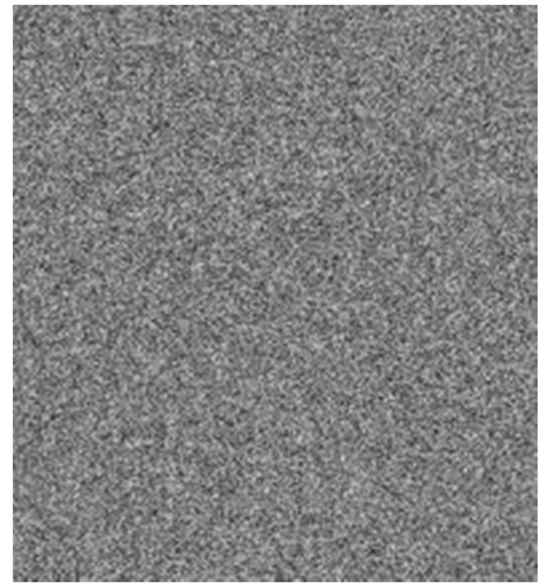
- Active attacks are easy ☹ (plaintext can be easily manipulated by removing, repeating, or interchanging blocks).

Encrypting bitmap images in ECB mode

original

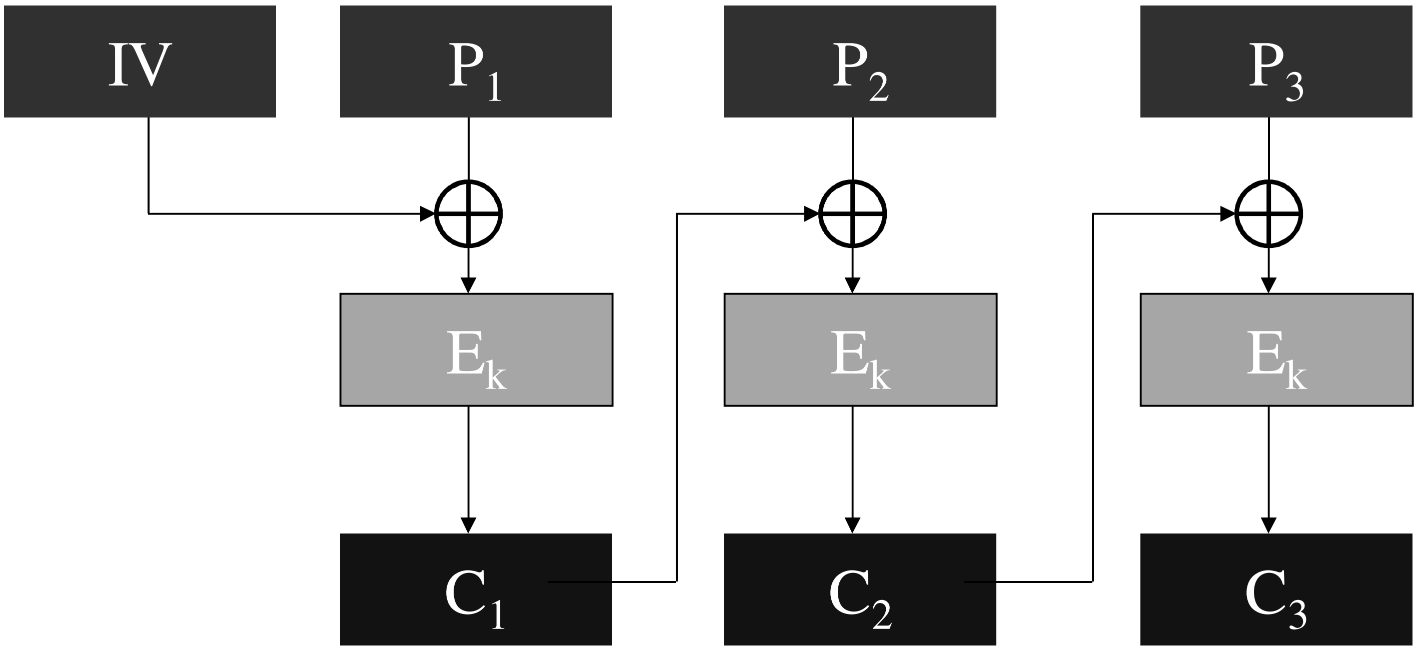


encrypted using ECB mode



encrypted using a secure mode

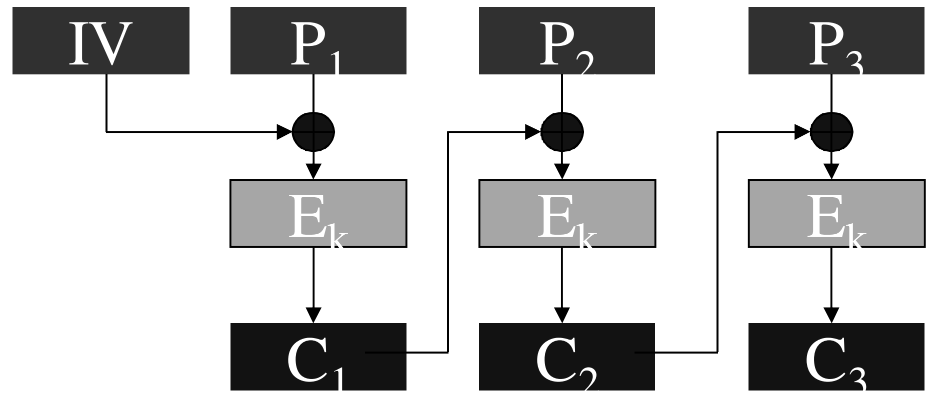
CBC Encryption Mode (Cipher Block Chaining)



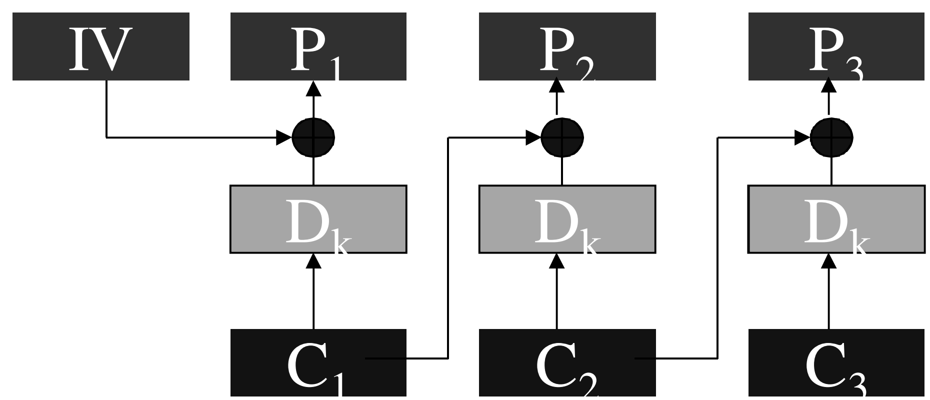
Previous *ciphertext* is XORed with current *plaintext* before encrypting current block.
An initialization vector IV is used as a "seed" for the process.
IV can be transmitted in the clear (unencrypted).

CBC Mode

Encryption:



Decryption:



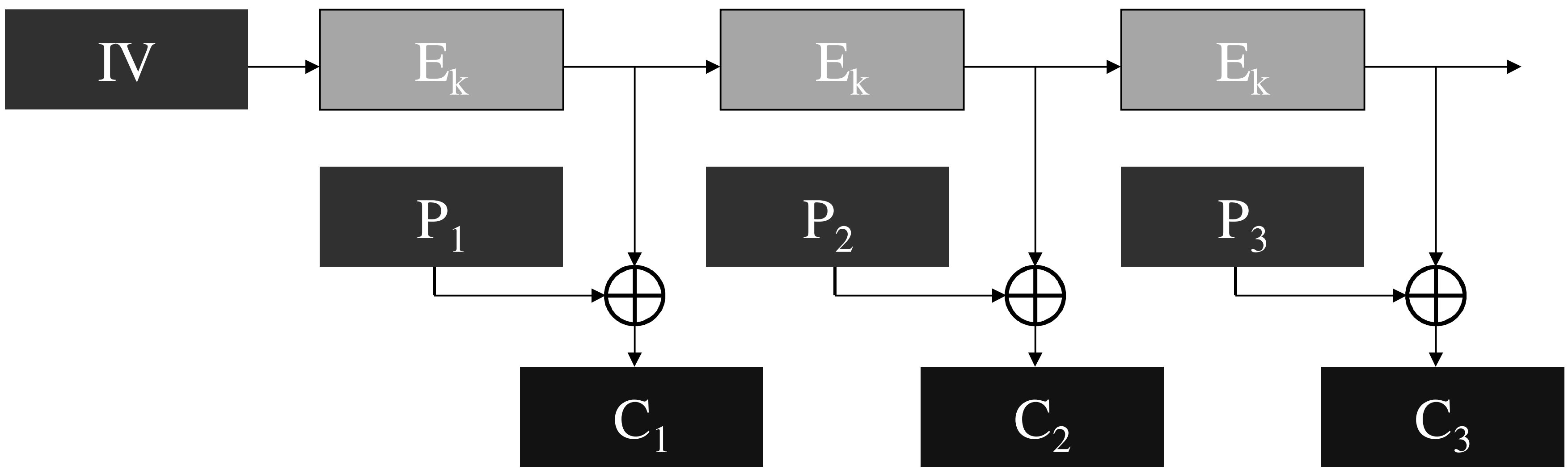
Properties of CBC

- Asynchronous: the receiver can start decrypting from any block in the ciphertext. ☺
- Errors in one *ciphertext* block propagate to the decryption of the next block (but that's it). ☺
- Conceals plaintext patterns (same block ⇒ different ciphertext blocks) ☺
  - If IV is chosen at random, and E_K is a pseudo-random permutation, CBC provides chosen-plaintext security.
  - But if IV is fixed, CBC does not even hide not common *prefixes.*
- No parallel implementation of encryption is known ☹
- Plaintext cannot be easily manipulated ☺
- Standard in most systems: SSL, IPSec, etc.

A chosen-plaintext attack on CBC if IV is known

- Suppose that adversary can predict IV for next message
  – Bug in SSL/TLS 1.0: IV for record #i is the last ciphertext block of record #(i-1)
- Attacker
  – Asks to receive encryption of X=0
    - Receives (IV', E(k, 0⊕IV')) = (IV', E(k,IV'))
  – Attacker knows IV for next ciphertext
  – Attacker can now distinguish between encryption of m_0=IV⊕IV' and any other m_1.
    - Encryption of m_0 is (IV, E(k, IV ⊕ (IV⊕IV'))) = (IV, E(k, IV'))

OFB Mode (Output FeedBack)



- An initialization vector IV is used as a "seed" for generating a sequence of "pad" blocks
  - $E_k(IV)$, $E_k(E_k(IV))$, $E_k(E_k(E_k(IV)))$,…
- Essentially a stream cipher.
- IV can be sent in the clear. Must never be repeated.

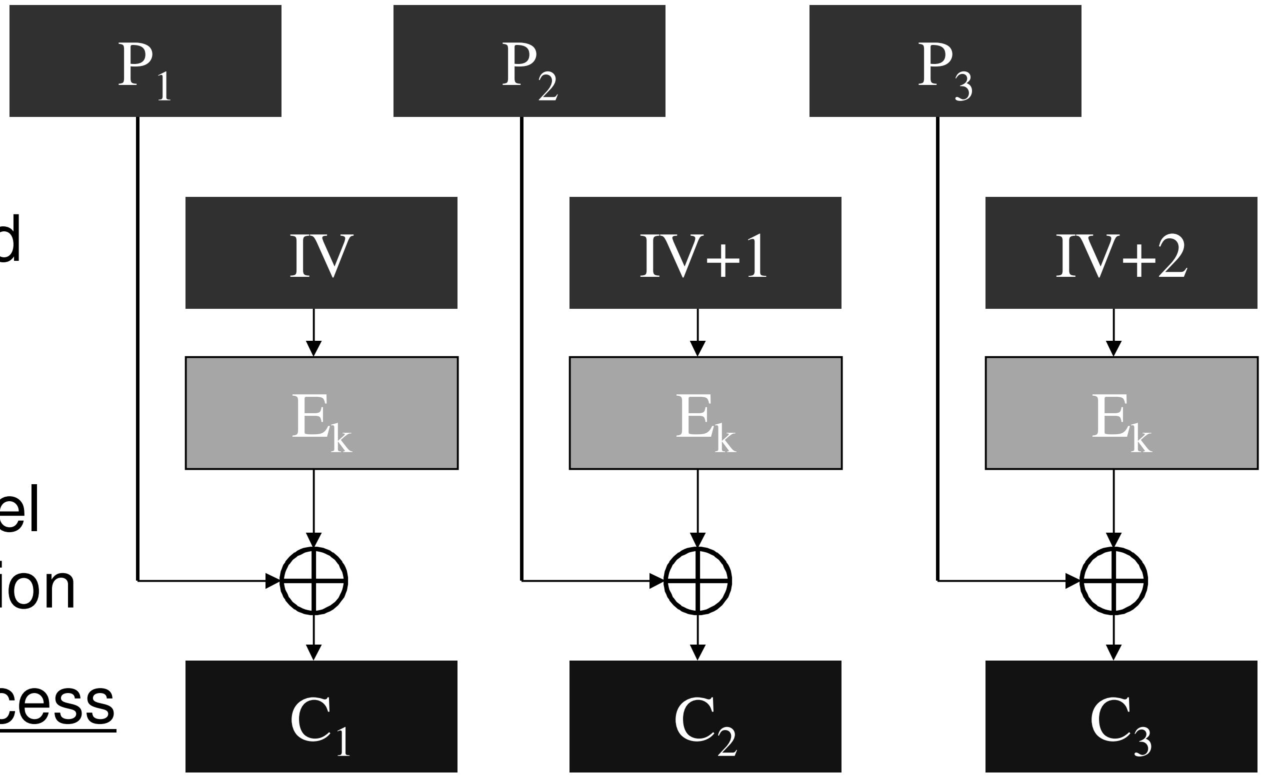
Properties of OFB

- Essentially implements a synchronous stream cipher. I.e., the two parties must know s_0 and the current bit position.
  - A block cipher can be used instead of a PRG.
  - The parties must synchronize the location they are encrypting/decrypting. ☹

- Conceals plaintext patterns. If IV is chosen at random, and E_K is a pseudo-random permutation, OFB provides chosen-plaintext security. ☺

- Errors in ciphertext do not propagate ☺
- Implementation:
  - Pre-processing is possible ☺
  - No parallel implementation is known ☹
- Active attacks (by manipulating the plaintext) are possible ☹

CTR (counter) Encryption Mode



IV is selected as a random value

- easy parallel implementation
- random access
- preprocessing
- no message expansion
- if E is a PRF then E_{CTR} provides chosen plaintext security

Pseudo-random functions

- A pseudo-random function is a function which cannot be distinguished from a random function.
  - The possible number of functions $f : \{0,1\}^n \rightarrow \{0,1\}^l$ is $2^{2^n l}$
  - A random function is one which is chosen at random from that range. Its representation must be at least $2^n l$ bits.
  - Alternatively, we can say that the random function chooses the value of $f(x)$ independently at random for every x.

Pseudo-random functions - definition

- $F : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}^*$
  - The first input is the key, and once chosen it is kept fixed.
  - For simplicity, assume $F : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^n$
  - $F(k,x)$ is written as $F_k(x)$

- F is pseudo-random if $F_k()$ (where k is chosen uniformly at random) is indistinguishable (to a polynomial distinguisher D) from a function *f* chosen at random from all functions mapping $\{0,1\}^n$ to $\{0,1\}^n$
  - There are 2^n choices of F_k, whereas there are $(2^n)^{2^n}$ choices for *f*.
  - The distinguisher D's task:
    - We choose a function G. With probability ½ G is F_k (where $k \in_R \{0,1\}^n$), and with probability ½ it is a random function *f*.
    - D can ask to compute $G(x_1),G(x_2),\ldots$ for any $x_1,x_2,\ldots$ it chooses.
    - D must then output 1 if $G=F_k$.
    - F_k is pseudo-random if $|Pr(D(F_k)=1)-Pr(D(f)=1)| \leq$ negligible.

Pseudo-random permutations

- $F_k(x)$ is a keyed permutation if for every choice of k, $F_k()$ is one-to-one.
  - Note that in this case $F_k(x)$ has an inverse, namely for every y there is exactly one x for which $F_k(x)=y$.

- $F_k(x)$ is a pseudo-random permutation if
  - It is a keyed permutation
  - It is indistinguishable (to a polynomial distinguisher D) from a permutation *f* chosen at random from all permutations mapping $\{0,1\}^n$ to $\{0,1\}^n$.
    - 2^n possible values for F_k
    - $(2^n)!$ possible values for a random permutation

A PRF can be used to construct a PRG

- Given a PRF F(k,x), $F : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^n$

The following $G:\{0,1\}^n \rightarrow \{0,1\}^{n \cdot t}$ is a secure PRG:

$$G(k) = F(k,0) \,||\, F(k,1) \,||\, \ldots \,||\, F(k,t-1)$$

(This is a parallelizable construction)

Proof: Suppose that an adversary can distinguish G(k) from a random string from $\{0,1\}^{n \cdot t}$. Then after asking to compute F(k,0),F(k,1),…,F(k,t) it can distinguish F() from a random function.

- Block ciphers are modeled as pseudo-random permutations.
- However, even a random permutation leaks some information if it is used to encrypt longer messages
  - Identical blocks result in identical ciphertexts.
- A stronger definition of security, and an appropriate construction are needed to prevent this information leakage.

CPA security of block ciphers

- CPA (chosen-plaintext attack) indistinguishability
  - A key k is chosen at random
  - The adversary is given access to $E_k()$, and can encrypt any message it wants.
  - The adversary *A* chooses two messages m_0,m_1.
  - A random message m_b is chosen, $b \in \{0,1\}$.
  - *A* is given a challenge ciphertext $E_k(m_b)$.
  - *A* can continue to compute $E_k()$ on any message.
  - *A* must output b'.
  - *A* succeeds if b=b'.
- The encryption scheme is (t,e)-CPA-secure if for all *A* that runs at most t steps, Pr(b=b') < 1/2+e.

Constructing CPA-secure encryption

- Note that the encryption must be probabilistic.

- Let $F\colon \{0,1\}^n \to \{0,1\}^n$ be a pseudo-random function.
- The construction
  - Choose a random key $k \in \{0,1\}^n$
  - Encryption of $m \in \{0,1\}^n$: choose random $r \in \{0,1\}^n$, output $c = (r, F_k(r) \oplus m)$.
  - Decryption of $c = (r, f)$: compute $m = F_k(r) \oplus f$.

  - Intuitively, $F_k(r)$ is indistinguishable from a random message, and therefore ciphertext is like a one-time pad.

Observations

- The encryption is probabilistic
- Encrypting the same message twice is likely to result in different ciphertexts, since different r values will be used.
- This is secure as long it is unlikely that the same value of r will be used twice.
- Instead of using a random r, one could use a nonce: a value that changes from message to message. For example, a counter.
- Ciphertext is longer than plaintext, since it must also include the randomness

Security

- Theorem: If F_k is a pseudo-random function then the encryption scheme is (t,ε)-CPA-indistinguishable.
- Proof sketch:
  - Lemma: If F_k is random, then the adversary can distinguish between $E(m_0),E(m_1)$ only if the challenge ciphertext is $(r, F_k(r) \oplus m_b)$, and r was used in one of the encryptions asked by the adversary.
    - The prob. of r being used in a previous encryption is $\leq t / 2^n$.
  - Proof: If r was not used in one of these encryptions than m_b is encrypted with a random one-time pad.
  - Replace the random function with a pseudo-random one.
    - Need to show that this change does not affect the probability of success in more than a negligible ε. (see next page)
  - Therefore total success probability is $< \frac{1}{2} + t/2^n + \varepsilon$.

Security (contd.)

Background:

- If F_k is random, then the adversary succeeds with prob $\leq t / 2^n$.

– Replace the random function with a pseudo-random F_k.

– Suppose that now success probability is > ½ + $t/2^n$ + p(n).

– Then we found a distinguisher D between F_k and a random function, which succeeds with prob > p(n).

- D has oracle access to a function G which is either random or is the prf F_k , and to an attacker A against the encryption.
- D constructs an encryption according to the construction, and lets A attack it. Whenever A asks for an encryption, D asks for a value of G and encrypts.
- If A succeeds in decryption, D claims that G is the prf. Otherwise D claims that G is random. $|Pr(D(Fk)=1)-Pr(D(G)=1)| = p(n) > neg.$